On the Almost Axisymmetric Flows with Forcing Terms

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Outline

- Analysis of the Hamiltonian of Almost Axisymmetric Flows.

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- A Toy Model.

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- A Toy Model.
- Challenges in the study of the Almost Axisymmetric Flows with Forcing Terms.

Time varying domain.

The time varying domain occupied by the fluid is given by

$$\Gamma_{r_1^t} := \{(\lambda, r, z) \,|r_0 \leq r \leq r_1^t(\lambda, z),\ z \in [0, H],\ \lambda \in [0, 2\pi]\},$$

For simplicity, we set $r_0 = 1$ in the sequel.

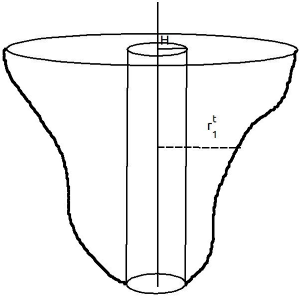

Figure: Time varying domain.

Hamiltonian

The fluid evolves with the velocity $\mathbf{u} := \mathbf{u}(\lambda, r, z)$ expressed in cylindrical coordinates (u, v, w) .
The temperature θ of the fluid inside the vortex is assumed to be greater that the ambient temperature maintained constant at $\theta_0 > 0$.

g is the gravitational constant.

The Hamiltonian of the Almost Axisymmetric Flow is

$$\int_{\Gamma_{r_1}} (\frac{u^2}{2} - g\frac{\theta}{\theta_0}) r dr dz d\lambda.$$

Important: The Almost Axisymmetric Flows are derived from Boussinesq's equations with no loss of the Hamiltonian structure (George Craig).

Hamiltonian : Stable Almost axisymmetric flows

Ω : Coriolis coefficient.

$ru + \Omega r^2$: angular momentum

$\frac{g}{\theta_0}\theta$: potential temperature.

Stability condition:
On each $\lambda-$ section of the domain Γ_{r_1}, we require that

$$(r, z) \longrightarrow [(ru^\lambda + \Omega r^2)^2, \frac{g}{\theta_0}\theta^\lambda]$$

be invertible and gradient of a convex function.

Hamiltonian: Stable Almost axisymmetric flows

We made crucial observation that, for stable Almost axisymetric flows for which the total mass is finite (=1), the Hamiltonian can be expressed in terms of one single measure σ:

$$\mathcal{H}[\sigma] = \int_0^{2\pi} I_0[\sigma_\lambda] + \inf_{\rho\in\mathcal{S}} I[\sigma_\lambda](\rho) d\lambda$$

Here, σ is a probability measure such that $\pi^1_{\#}\sigma$ is absolutely continuous with respect to $\mathcal{L}^1_{|[0,2\pi]}$.

$$I_0[\sigma_\lambda] = \int_{\mathbb{R}^2_+} \left(\frac{y_1}{2} - \Omega\sqrt{y}_1 - \frac{|y|^2}{2} \right) \sigma_\lambda(dy)$$

$$I[\sigma_\lambda](\rho) := \frac{1}{2} W_2^2\Big(\sigma_\lambda, \frac{1}{(1-2x_1)^2}\chi_{D_\rho(x)}\Big) + \int_{D_\rho} \Big(\frac{\Omega^2}{2(1-2x_1)} - \frac{|x|^2}{2} \Big) \frac{1}{(1-2x_1)^2} dx$$

Here, $\mathcal{S}$ is the set of functions $\rho : [0, H] \to [0, 1/2)$,

$$D_\rho := \{x = (x_1, x_2) \mid x_1 \in [0, H], 0 \le x_2 \le \rho(x_1)\}$$

Analysis of the Hamiltonian

Assume σ_0 is a probability measure on $\mathbb{R}^2$ and write

$$I[\sigma_0](\rho) = \frac{1}{2} W_2^2\left(\sigma_0, \frac{1}{(1-2x_1)^2}\chi_{D_\rho}(x)\right) + \text{good terms}$$

Existence of a minimizer.
Obstacle : $\left\{\chi_{D_\rho}\right\}_{\rho \in \mathcal{S}}$ is not weakly* closed in L^∞.

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However,

$$I[\sigma_0](\rho^{\#}) \leq I[\sigma_0](\rho)$$

where $\rho^{\#}$ is the increasingly monotone rearrangement of ρ.
Classical results in the direct methods of the calculus of variations ensures the existence of a minimizer.

Analysis of the Hamiltonian

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We use a Dual formulation of the minimization problem that yields existence and uniqueness.

$$\sup_{\{(P,\Psi):P=\Psi^*,\Psi=P^*\}} \int_{\mathbb{R}^2} \left(\frac{y_1}{2} - \Omega\sqrt{y}_1 - \Psi(y) \right) \sigma_0(dy) + \inf_{\rho\in\mathcal{S}} \int_0^H \Pi_P(\rho(x_2), x_2) dx_2 \tag{1}$$

$$\Pi_P(x_1, s) = \int_0^s \Big(\frac{1}{2(1-2x_1)} - P(x_2, x_1) \Big) \frac{1}{(1-2x_2)^2} dx_1 \quad \text{for} \quad 0 \le x_1 < 1.$$

(1) has a unique solution.

Analysis of the Hamiltonian.

Regularity of the boundary ∂D_ρ

The dual problem reveals a regularity property of ρ stronger than monotonicity.

More precisely, if $\mathrm{spt}(\sigma_0) \subset (\frac{1}{L_0}, L_0) \times (0, L_0)$ $L_0 > 0$ and P^{σ_0} solve the variational problem (1) then the study of Euler -Lagrange equation of

$$\inf_{\rho \in \mathcal{S}} \int_0^H \Pi_{P^{\sigma_0}}(\rho(x_2), x_2) dx_2$$

yields $C > 0$ such that the minimizer ρ^{σ_0} satisfies

$$\rho^{\sigma_0}(\bar{x}_2) - \rho^{\sigma_0}(x_2) \geq C(\bar{x}_2 - x_2)$$

for all $x_2, \bar{x}_2 \in [0, H]$. Consequently, we obtain that $\partial D_{\rho^{\sigma_0}}$ is piecewise Lipschitz continuous.

A unusual Monge-Ampère equation.

Moreover, assume in addition, σ_0 is absolutely continuous with respect to the Lebesgue measure.
If $(P^{\sigma_0}, \Psi^{\sigma_0}, \rho^{\sigma_0})$ is the variational solution(1) then P^{σ_0} is convex, ∇P^{σ_0} is invertible $(1-2x_1)^{-2}\chi_{D_\rho}(x)\mathcal{L}^2$ a.e and

$$\begin{cases} (i) & \frac{1}{(1-2\partial_{y_2}\Psi)^2}\det \nabla^2\Psi = \sigma_0 \\ (ii) & P\Big(\rho(x_2), x_2\Big) = \frac{\Omega^2}{2(1-2\rho(x_2))} \quad \text{on } \{\rho > 0\} \\ (iii) & \nabla\Psi \quad \text{maps} \quad spt(\sigma_0) \quad \text{onto } D_\rho. \end{cases} \tag{2}$$

Change of variables

Let $(P_\lambda, \Psi_\lambda, \rho_\lambda)$ be the solution to the variational problem (1) corresponding to σ_λ. Assume σ absolutely continuous with respect to Lebesgue.

Define u, θ, r through

$$(u_\lambda r + \Omega r^2)^2 = \partial_{x_1} P_\lambda, \quad g\frac{\theta_\lambda}{\theta_0} = \partial_{x_2} P_\lambda, \quad 2x_1 = 1 - r^{-2}. \tag{3}$$

and

$$\chi_{\Gamma_{r_1}} r dr dz d\lambda = (1-2x_1)^{-2} \chi_{D_{\rho_\lambda}}(x) dx_1 dx_2 d\lambda = \sigma dy_1 dy_2 d\lambda.$$

Then, (u, θ, r_1) satisfy the stability condition and

$$\mathcal{H}[\sigma] = \int_{\Gamma_{r_1}} (\frac{u^2}{2} - g\frac{\theta}{\theta_0}) r d\lambda dr dz.$$

Forced Axisymmetric Flows : Toy Model 2D

We remove the λ dependence on the quantities involved in the Almost axisymmetric flows with forcing terms to obtain the forced axisymmetric flows: $\frac{D}{Dt} := \partial_t + v\partial r + w\partial z$.

$$\begin{cases} (ru + \Omega r^2)^2 = r^3 \partial_r[\varphi + \frac{\Omega^2}{2} r^2], \frac{g}{\theta_0}\theta = \partial_z[\varphi + \frac{\Omega^2}{2} r^2] \text{ in } \Gamma_{r_1} \\ \frac{1}{r}\partial_r(rv) + \partial_z w = 0 \quad \text{in } \Gamma_{r_1} \\ \partial_t r_1 + w\partial_z r_1 = v, \quad \text{on } \{r = r_1\} \\ \frac{D}{Dt}(ru + \Omega r^2) = F, \quad \frac{\bar{D}}{Dt}(\frac{g}{\theta_0}\theta) = \frac{g}{\theta_0} S \quad \text{in } \Gamma_{r_1} \end{cases} \tag{4}$$

Here,

$$\Gamma_{r_1^t} := \{(r, z) \ | \ r_1(t, z) \geq r \geq r_0, \ z \in [0, H]\},$$

$$\varphi(t, r_1(t, z), z) = 0, \quad \text{on} \quad \partial\{r_1 > r_0\}. \tag{5}$$

Neumann condition has been imposed on the rigid boundary.

Data : F, S are prescribed functions.

Unknown :u, v, w, φ, θ and r_1

Toy Model in "Dual Space" $2D$

In view of the change of variable discussed above, existence of a variational solution to the MA equation, formal computations yield

- $$\text{Toy Model} \Longleftrightarrow \begin{cases} \partial_t \sigma_t + div(\sigma_t V_t[\sigma_t]) = 0 \\ \sigma_{|t=0} = \bar{\sigma}_0 \end{cases}$$

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- <u>Task we completed:</u>

  Identify the operator $\sigma \longmapsto V_t[\sigma]$.

Forced axisymmetric flows: Velocity field

Regular initial data:

$$V_t[\sigma](y) = \mathbb{L}_t(\nabla\Psi^\sigma(y); y)$$

where

$$\mathbb{L}_t(x; y) = \Big(2\sqrt{y}_1 F_t\big((1-2x_1)^{-\frac{1}{2}}, x_2\big), \frac{g}{\theta_0} S_t\big((1-2x_1)^{-\frac{1}{2}}, x_2\big)\Big).$$

and

Ψ^σ is a solution in the variational problem (1).

General initial data:
Use the Riesz representation theorem to uniquely define $V_t[\sigma]$ by

$$\int_{\mathbb{R}^2} \langle V_t[\sigma], G\rangle d\sigma = \int_{D^\sigma_{\rho^\sigma}} e(x_1)\langle \mathbb{L}_t(x, \nabla P^\sigma), G(\nabla P^\sigma)\rangle dx_1 dx_2$$

$\forall G \in C_c(\mathbb{R}^2, \mathbb{R}^2)$ and (P^σ, ρ^σ) solves the variational problem (1).

Existence of solutions for the Forced axisymmetric flows.

- Appropriate conditions of the forcing terms.
- Continuity property in $\sigma \longrightarrow V_t[\sigma]$ (and $\sigma \longrightarrow \sigma V_t[\sigma]$).

$\Longrightarrow$ Global solution in time.

Almost Axisymmetric Flow with Forcing Terms

Back to the full physical model
These equations are given by (here, $\frac{D}{Dt} := \partial_t + \frac{u}{r}\partial_\lambda + v\partial r + w\partial z$)

$$\begin{cases} r\left(\frac{Du}{Dt} + \frac{uv}{r} + \frac{1}{r}\partial_\lambda\varphi + 2\Omega v\right) = F, \qquad \frac{u^2}{r} + 2\Omega u = \partial_r\varphi, \qquad \frac{D\theta}{Dt} = S, \\ \frac{1}{r}\partial_r(rv) + \frac{1}{r}\partial_\lambda u + \partial_z w = 0 \qquad \partial_z\varphi - g\frac{\theta}{\theta_0} = 0 \\ \partial_t r_1 + \frac{u}{r_1}\partial_\lambda r_1 + w\partial_z r_1 = v \text{ on } \{r = r_1\} \end{cases} \tag{6}$$

in the region

$$\Gamma_{r_1} := \{(\lambda, r, z) \mid r_1(\lambda, z) \geq r \geq r_0, \ z \in [0, H], \ \lambda \in [0, 2\pi]\},$$

subject to the boundary condition

$$\varphi(t, \lambda, r_1(t, \lambda, z), z) = 0, \quad \text{on} \quad \partial\{r_1 > r_0\}. \tag{7}$$

Neumann condition has been imposed on the rigid boundary.

Almost axisymmetric Flow with Forcing Terms : Dual space $3D$

The equations above can be recast as a transport equation :

$$\partial_t \sigma_t + div(\sigma_t X_t[\sigma_t]) = 0; \qquad \sigma_{|t=0} = \bar{\sigma}_0 << \mathcal{L}^3 \tag{8}$$

Here

$$X_t[\sigma](y) = \mathbb{L}_t(\nabla \Psi^\sigma(y), y)$$

$\Psi^\sigma(\lambda, \cdot)$ solves the Monge Ampère equations (2)
and

$$\mathbb{L}_t(x, y) =$$

$$\left(\frac{\sqrt{y_1}}{r_0} - \Omega - 2x_1\sqrt{y_1}, 2\sqrt{y_1} F_t(\lambda, e^{\frac{1}{4}}(x_1), x_2) + 2x_1\sqrt{y_1}, \frac{g}{\theta_0} S_t(\lambda, e^{\frac{1}{4}}(x_1), x_2) \right)$$

with $x = (\lambda, x_1, x_2)$, $y = (\lambda, y_1, y_2)$ and $e(x_1) = (1 - 2x_1)^{-2}$.

Challenges in the continuity equation

- Defining well the velocity $X_t[\sigma]$.
- Existence and Regularity of

$$\nabla\Psi = \left(\frac{\partial\Psi}{\partial\lambda}, \frac{\partial\Psi}{\partial\Upsilon}, \frac{\partial\Psi}{\partial Z}\right)$$

- Regularity in a Monge-Ampere equation with respect to a parameter:

$$\frac{1}{(1-2\partial_{y_1}\Psi^\lambda)^2}\det\nabla^2_{y_1,y_2}\Psi^\lambda = \sigma^\lambda$$

Thank you for your attention!