Relaxed Lagrangian solutions for the Semi-Geostrophic Shallow Water system in physical space

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Based on joint work with M. Feldman (UW-Madison)

Setting

- SGSW models the motion of a fluid rapidly rotating around the vertical axis x_3, contained within the evolving 3-dimensional region $\mathcal{D}(t)$ which has the structure:

$$\mathcal{D}(t) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ (x_1, x_2) \in \Omega, \ 0 \leq x_3 \leq h(t, x_1, x_2)\},$$

where the region Ω of the (x_1, x_2)-plane is given and fixed, but the height h above the reference level is unknown and can evolve in time. The pressure on the top boundary of the fluid is a given constant p_0, and

$$p(t, x_1, x_2, x_3) = [h(t, x_1, x_2) - x_3] + p_0.$$

- The horizontal components of the velocity $\mathbf{v} = (\mathbf{u}, u_3)$ of the fluid are independent of x_3.

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SGSW in terms of the modified pressure

- Introduce
$$P(t,x) = p(t,x) + (x_1^2 + x_2^2)/2,$$
to write SGSW as
$$\begin{aligned}
&D_t X = J\big(X - x\big), \\
&\partial_t h + \nabla \cdot (h\mathbf{u}) = 0 \\
&X = \nabla P,\ P = h + \frac{1}{2}|\mathrm{Id}_\Omega|^2 && \text{in } [0,T) \times \Omega; \\
&\mathbf{u} \cdot \nu = 0 && \text{on } [0,T) \times \partial\Omega, \\
&P(0,\cdot) = P_0 && \text{in } \Omega,
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- Here $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

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Cullen-Purser stability

- Let $(P, \mathbf{u})$ be a solution and set $\mathcal{X}(t) := \{(Y, \rho) \ : \ Y : \Omega \to \mathbb{R}^2$ Borel, $\rho \in \mathcal{P}^{ac}(\Omega)$, $Y_{\#}\rho = \nabla P(t,\cdot)_{\#}h(t,\cdot)\}$.
- (Cullen & Shutts) Then $(\nabla P(t,\cdot), h(t,\cdot))$ is a critical point for

$$I(Y,\rho) = \int_\Omega |Y(x) - x|^2 \rho(x)dx + \int_\Omega \rho^2(x)dx$$

over $\mathcal{X}(t)$, for all $t \in (0,T)$.
- (Cullen & Purser) Only minimizers are stable, in the sense that SG accurately describes their evolution.
- In the language of Optimal Transport, this means $X(t,\cdot)$ must be the gradient of a convex function, i.e. $P(t,\cdot)$ must be convex for all $t \in (0,T)$.
- Thus, one is only interested in solutions satisfying this constraint.

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Do we have solutions?

- No existence results for weak Eulerian solutions.

Theorem (Feldman & T.)

Let $(P, \mathbf{u})$ be a distributional solution for SG in the physical space such that $\nabla P \in H^1(0, T; L^2(\Omega; \mathbb{R}^3))$. Then $\alpha_t := \nabla P_{t\#}\chi$ is atom-free for $\mathcal{L}^1$–a.e. $t \in (0, T)$.

- (Cullen) The model must accommodate solutions for which ∇P_t is locally constant. Observations show that the atmosphere contains significant regions where the potential temperature and absolute momentum of the atmosphere are well-mixed, representing a state of neutrality to parcel displacements (yielding "flat spots" in ∇P_t). Such states commonly arise as a result of atmospheric forcing either through surface heating or latent heat release.

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A change of variable and a new system

- Let $\nabla P_{t\#}h_t =: \alpha_t$. If $\int_\Omega h_0(x)dx = 1$, then α_t is a Borel probability.
- Let $X = \nabla P(t,x)$. Then α_t solves:

$$\partial_t \alpha + \nabla \cdot (U\alpha) = 0 \quad \text{in } [0,T) \times \mathbb{R}^2;$$
$$\nabla P(t,\cdot)_{\#} h(t,\cdot) = \alpha(t,\cdot) \quad \text{for any } t \in [0,T);$$
$$P(t,x) = h(t,x) + |x|^2/2, \quad \text{with } P(t,\cdot) \text{ convex for all } t \in [0,T);$$
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where $\bar{\gamma}(t,X)$ denotes the barycentric projection onto α_t of the optimal transfer plan between α_t and h_t, i.e.

$$\int_{\mathbb{R}^2} \xi(X) \cdot \bar{\gamma}(X)\, \alpha(dX) = \iint_{\mathbb{R}^2 \times \Omega} \xi(X) \cdot y\, \gamma(dX, dy)$$

for all continuous $\xi : \mathbb{R}^2 \to \mathbb{R}^2$ of at most quadratic growth, where γ is the (unique) optimal plan between α and h.

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SG in dual variables

- Existence of solutions in dual variables: Cullen & Gangbo in the case $\alpha_t \ll \mathcal{L}^2$.
- This is a Hamiltonian system (Gangbo & Pacini, Ambrosio & Gangbo) for

$$H(\mu) = -\frac{1}{2} \inf_{h \in \mathcal{P}^{ac}(\Omega)} \{W_2^2(\mu, h) + \|h\|^2_{L^2(\Omega)}\}.$$

- Uniqueness is open!

Formal Lagrangian flow

- Assume one has a classical solution to SGSW in physical space.
- If $\mathbf{u}$ has a flow $F : [0,T] \times \Omega \to \Omega$ defined by

$$\dot{F}(t,x) = \mathbf{u}(t, F(t,x)), \quad F(0,x) = x \text{ for } h_0\text{−a.e. } x \in \Omega,$$

then F can replace $\mathbf{u}$ as an unknown. Let $Z(t,x) := \nabla P(t, F(t,x))$.

- The system for (P, F) becomes

$$\begin{array}{ll} \partial_t Z(t,x) = J\big[Z(t,x) - F(t,x)\big] & \text{for } t \in [0,T) \text{ and } h_0\text{−a.e. } x \in \Omega, \\ Z(0,x) = \nabla P_0(x) & h_0 - \text{a.e. in } \Omega. \end{array} \qquad \text{(LE)}$$

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Definition of weak Lagrangian solutions

Let $P_0 \in L^\infty(\Omega)$ be convex such that $h_0 := P_0 - |\mathrm{Id}_\Omega|^2/2 \in \mathcal{P}(\Omega)$, and let $p \in [1, \infty)$. Let $P : [0, T) \times \Omega \to \mathbb{R}$ such that

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Let $F \in C([0, T); L^p(h_0; \mathbb{R}^2))$ be a Borel map. The pair (P, F) is called a weak Lagrangian solution of SGSW in physical space if

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- for any $t > 0$ we have $F_{t\#} h_0 = h_t$;
- There exists a Borel map $F^* : [0, T) \times \Omega \to \Omega$ such that for every $t \in (0, T)$ we have $F^*_{t\#} h_t = h_0$, and $F^*_t \circ F_t(x) = x$ for h_0-a.e. $x \in \Omega$ and $F_t \circ F^*_t(x) = x$ for h_t-a.e. $x \in \Omega$;
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Let $F \in C([0, T); L^p(h_0; \mathbb{R}^2))$ be a Borel map. The pair (P, F) is called a weak Lagrangian solution of SGSW in physical space if

- $F(0, \cdot) = \mathrm{Id}_\Omega \; h_0-$ a.e. in Ω, $P(0, x) = P_0(x)$ for a.e. $x \in \Omega$,
- for any $t > 0$ we have $F_{t\#} h_0 = h_t$;
- There exists a Borel map $F^* : [0, T) \times \Omega \to \Omega$ such that for every $t \in (0, T)$ we have $F^*_{t\#} h_t = h_0$, and $F^*_t \circ F_t(x) = x$ for h_0-a.e. $x \in \Omega$ and $F_t \circ F^*_t(x) = x$ for h_t-a.e. $x \in \Omega$;
- Lagrangian equation (LE) is satisfied in the sense of distributions.

Definition of weak Lagrangian solutions

Let $P_0 \in L^\infty(\Omega)$ be convex such that $h_0 := P_0 - |\mathrm{Id}_\Omega|^2/2 \in \mathcal{P}(\Omega)$, and let $p \in [1, \infty)$. Let $P : [0,T) \times \Omega \to \mathbb{R}$ such that

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Consistency and existence

- Cullen & Feldman proved that by setting $\mathbf{u}(t,x) := \partial_t F(t, F^*(t,x))$ under the assumption $\partial_t F \in L^\infty((0,T)\times\Omega;\mathbb{R}^2)$, we obtain that the pair $(P,\mathbf{u})$ is a weak (Eulerian) solution of SGSW in physical space.
- Cullen & Feldman used Ambrosio's theory of regular Lagrangian flows to construct weak Lagrangian solutions in physical space in the case $\alpha_0 =: \nabla P_{0\#} h_0 \in L^p(\nabla P_0(\Omega))$ for some $p > 1$, where P_0 is bounded and convex in an open ball containing $\bar{\Omega}$.
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When α_0 is singular; example

- If $\alpha_0 = 1.25\,\pi\delta_{z_0}$ for some $z_0 \in \partial B(0,1)$, one can readily check that if $\dot{z}(t) = 0.8\,Jz(t),\ z(0) = z_0$, then $\alpha_t := 1.25\,\pi\delta_{z(t)}$ solves SGSW in dual space.
- Here, $h_t(x) = 2 - |x - z(t)|^2/2$, $P_t(x) = x \cdot z(t) + 3/2$.
- There are no maps F_t as in the definition of weak Lagrangian solutions.
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An observation (1)

- For $\xi \in C^1(\mathbb{R}^2) \cap \mathrm{Lip}(\mathbb{R}^2)$ the map $t \mapsto \xi(Z(t,x))$ is absolutely continuous for $\mathcal{L}^2$-a.e. $x \in \Omega$ and

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for a.e. $t \in [0,T]$. Consequently, a more general, "renormalized" version of (LE) is available in the form

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Relaxed notion; definition

- Let $P_0 : \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$ be convex such that $P_0|_\Omega \in L^2(\Omega)$, $P_0|_\Omega - \frac{1}{2}|\mathrm{Id}_\Omega|^2 =: h_0 \in \mathcal{P}(\Omega)$ and $\nabla P_0|_\Omega \in L^2(h_0; \mathbb{R}^2)$ (C)

Definition of Relaxed (Renormalized) Lagrangian Solutions

Consider a Borel function $P : [0,T) \times \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$ such that $P(t,\cdot)$ is convex for all $t \in [0,T)$ and a Borel family of probability measures $[0,T) \ni t \mapsto \sigma_t \in \mathcal{P}(\Omega \times \Omega)$. Let σ be given by $d\sigma = d\sigma_t\, dt$. We say that (P,σ) is a *relaxed Lagrangian solution* for the SGSW system with initial data P_0 if

1. $P(0,\cdot)|_\Omega \equiv P_0|_\Omega$;
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Weak stability

Theorem

Let $P_0,\ P_0^n$ satisfy (C) for all positive integers n with respect to Ω. Assume (P^n, σ^n) are relaxed solutions for SGSW in physical space corresponding to the initial data P_0^n. Then, possibly up to a subsequence, (P^n, σ^n) converges to a relaxed solution (P, σ) corresponding to the initial datum P_0. The convergence is in the following sense:

(i) $P_t^n \rightharpoonup P_t$ weakly in $L^2(\Omega)$ and locally uniformly in Ω for all $t \in [0, T]$;

(ii) $\nabla P_t^n \to \nabla P_t$ a.e. in Ω for all $t \in [0, T]$, and $\{\nabla P_t\}_n$ is locally bounded uniformly with respect to $t \in [0, T]$ and $n \geq 1$;

(iii) σ^n converges narrowly to σ.

Furthermore, the corresponding dual space solutions satisfy $W_p(\alpha_t^n, \alpha_t) \to 0$ for all $t \in [0, T]$ and all $1 \leq p < 2$.

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Existence

Corollary

Let $\Omega \subset \mathbb{R}^2$ be open, bounded and connected. Let P_0 satisfy (C) with respect to Ω. Then there exists a relaxed Lagrangian solution for SGSW corresponding to the initial data P_0.

A few important observations

- Renormalization is used to show that the relaxed solutions give rise to dual-space solutions.
- The variational characterization of P, h, α was extended (with stability) from the case $\alpha \ll \mathcal{L}^2$ (Cullen & Gangbo) to $\alpha \in \mathcal{P}_2(\mathbb{R}^2)$: h minimizes $\mathcal{P}^{ac}(\Omega) \ni \rho \mapsto W_2^2(\rho, \alpha) + \|\rho\|^2_{L^2(\Omega)}$ iff $P = h + |\mathrm{Id}_\Omega|^2/2$ satisfies (C) and $\alpha = \nabla P_{\#} h$.
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Thank you!