On Equi-/Over-/Underdispersion and Related Properties of Some Classes of Probability Distributions

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Definition 1 *(Lambert W function and its principal branch W_0, see [1]).*

(i) *Complex-valued Lambert function $W(z)$ is defined as the multi-valued inverse of function $y(x) := x \cdot e^x$. Equivalently, it can be defined as the function satisfying the identity*

$$W(z) \cdot e^{W(z)} \equiv z,$$

where $z \in \mathbb{C}$.

Its Taylor series around $z = 0$,

$$W(z) = \sum_{\ell=1}^{\infty} w_\ell \cdot z^\ell, \tag{1}$$

has the radius of convergence $1/e$.

The coefficients $\{w_\ell$'s, $\ell \in \mathbf{N}\}$ are as follows:

$$w_\ell = (-\ell)^{\ell-1}/\ell! \tag{2}$$

(ii) *The series (1)–(2) can be extended to a holomorphic function on* $\mathbb{C}$ *with a branch cut along* $(-\infty, -1/e]$. *This function defines the principal branch* $W_0(z)$ *of* $W(z)$.

Definition 2 *(Index of dispersion or the variance-to-mean ratio). Given r.v.* $\mathcal{Y}$ *with finite variance, its index of dispersion which is hereinafter denoted by* **VMR**, *is defined as follows:*

$$\mathbf{VMR}(\mathcal{Y}) := \mathbf{Var}(\mathcal{Y})/\mathbf{E}(\mathcal{Y}).$$

Index of dispersion for Poisson distribution $= 1$, but its values for binomial and negative binomial distributions are < 1 & > 1, respectively.

It is *overdispersion* which is exhibited more frequently by the data.

A toy example of distribution theory: Neyman Type A EDM and Lambert W function

The additive Neyman type A exponential dispersion model is comprised of non-negative infinitely divisible distributions on $\mathbf{Z}_+$ such that its generic member is a compound Poisson sum of i.i.d. Poisson-distributed r.v.'s as well as the Poisson mixture with Poisson mixing measure.

This EDM can be constructed starting from its member $\mathcal{X}$ whose c.g.f. $\{\Psi_{\mathcal{X}}(v), v \in \mathbf{R}^1\}$ is as follows:

$$\Psi_{\mathcal{X}}(v) := \log \mathbf{E} e^{v\mathcal{X}} = e^{e^v - 1} - 1.$$

Theorem 3 *(see [5, Th. 5.1]). The u.v.f. $\mathbf{V}_{\mathcal{X}}(\mu)$ of the Neyman type A EDM which is constructed starting from r.v. $\mathcal{X}$ has domain $\mathbf{R}^1_+$, where it is expressed as follows:*

$$\mathbf{V}_{\mathcal{X}}(\mu) = \mu \cdot (1 + W_0(e \cdot \mu)). \qquad (3)$$

Since $W_0(x) > 0$ for real $x > 0$, **(3) implies overdispersion automatically.**

Also, since

$$W_0(z) \sim z \;\text{ as } z \downarrow 0;$$

$$W_0(z) \sim \log z \;\text{ as } z \to +\infty,$$

a combination of (3) with [2, Ch. 4] and [3, pp. 410–411] implies that this EDM is *locally Poisson*, both at 0 and at $+\infty$. In particular, as $\mu \to +\infty$,

$$\mathbf{V}_{\mathcal{X}}(\mu) \sim \mu \cdot \log\ \mu \tag{4}$$

[3, pp. 410–411] provides several assertions on weak convergence to members of the *power-variance family* under assumptions on *regular variation* of u.v.f. However, [3] did not provide specific examples which involve a non-trivial *regularly varying* function per se – all the illustrative examples therein concern just power functions. The following result fills in this gap.

Corollary 4 *(see [5, p. 2040]). A combination of [3, pp. 410–411] with (4) implies that for an arbitrary fixed* $\mu \in \mathbf{R}^1_+$,

$$\frac{1}{\log c} \sum_{i=1}^{Tw_1(\mu,1)} Tw_1^{(i)}(\log c, 1) \xrightarrow{\mathrm{d}} Tw_1(\mu, 1) \quad (5)$$

as $c \to +\infty$.

Here, $Tw_1(\mu, 1)$ is Poisson r.v. with mean μ. It is relevant that it is an application of [3, pp. 410–411] which necessitates the use of the slowly varying function $\log c$ for normalizing purposes on l.h.s. of (5). (The power index = 0.) This is is parallel to classical limit theorems on general domains of attraction to stable distributions. At the same time, it is evident that (5) can be rewritten as follows:

$$\frac{1}{b} \sum_{i=1}^{\mathcal{P}oiss(\mu)} \mathcal{P}oiss^{(i)}(b) \xrightarrow{\mathrm{d}} \mathcal{P}oiss(\mu) \qquad (6)$$

as $b \longrightarrow +\infty$, which can also be established by the method of m.g.f.'s. Note that (6) is **NOT** a result of Poisson law of small numbers type, but that on a cluster structure evolution!

Other overdispersed non-negative integer-valued distributions which are related to Poisson

1) zero-modified Poisson; parameter $\delta \in (0, 1)$:

$$p_0 = \delta + (1 - \delta) \cdot e^{-\mu};$$

$$p_n = (1 - \delta) \cdot e^{-\mu} \cdot \frac{\mu^n}{n!}, \quad n \geq 1.$$

2) generalized Poisson (or back-shifted Borel) distribution which emerges, among other things, as the law of the total progeny of a Galton-Watson branching process in the case where the mechanism of local branching is Poisson with mean ≤ 1.

Special case:

$$p_n = e^{-(n+1)} \cdot \frac{(n+1)^{n-1}}{n!}, \quad n \geq 0.$$

Its p.g.f. is expressed in terms of Lambert W_0 function.

Extended family of zero-modified geometric distributions

Definition 5 *For* $\gamma \in (0,1)$, $r \in [-(1-\gamma)/\gamma, 1)$, *non-negative integer-valued r.v.* $Y_{\gamma,r} \in EFZMGL$ *if*

$$\mathbf{P}\{Y_{\gamma,r} = 0\} = \gamma,$$

and $\forall\ k \in \mathbf{N}$,

$$\mathbf{P}\{Y_{\gamma,r} = k\}$$
$$= \gamma(1-\gamma)(1-r)\left\{1-\gamma+\gamma r\right\}^{k-1}.$$

Special cases: *(i)* $Y_{\gamma,0}$ - standard geometric;

$$(ii) \quad Y_{\gamma,-(1-\gamma)/\gamma} \stackrel{\mathrm{d}}{=} \mathbf{B}(1, 1-\gamma).$$

The mean, variance, skewness and kurtosis are all available in the closed form.

Shannon entropy:

$$\mathbf{H}_\gamma(r) := -\sum_{k=0}^{\infty} \mathbf{P}\{Y_{\gamma,r} = k\} \log_2 \mathbf{P}\{Y_{\gamma,r} = k\}$$
$$= -\{\gamma \cdot \log_2 \gamma + (1-\gamma) \cdot \log_2(1-\gamma)\}$$
$$-\frac{1-\gamma}{\gamma(1-r)}\{(1-\gamma+\gamma r) \cdot \log_2(1-\gamma+\gamma r)$$
$$+\gamma(1-r) \cdot \log_2(\gamma(1-r))\}.$$

This formula is consistent with already known expressions for Shannon entropy of Bernoulli r.v. $Y_{\gamma,-(1-\gamma)/\gamma}$, which is frequently termed binary entropy function, and also of geometric r.v. $Y_{\gamma,0}$.

We decompose EFZMGL into separate NEFs. To this end, consider the following quantity, which turns out to be an invariant of the exponential tilting transformation:

$$\mathcal{I}_{\gamma,r} := \frac{1-\gamma+\gamma r}{(1-\gamma)(1-r)} \in [0,+\infty). \qquad (7)$$

In particular, $\mathcal{I}=0$ and 1 correspond to Bernoulli and geometric NEFs, respectively.

$(\phi, \mathcal{I})$**-parameterization.** $\forall \mathcal{I} \in \mathbf{R}^1_+$, define counting measure $\nu_{\mathcal{I}}(\{k\})$ on $\mathbf{Z}_+$ such that

$$\nu_{\mathcal{I}}(\{k\}) = \begin{cases} 1 & \text{if } k=0, \\ 1/\mathcal{I} & \text{if } k \geq 1. \end{cases}$$

Let canonical parameter $\phi \in \Phi = (-\infty, 0)$.

For such ϕ's, we introduce cumulant

$$\begin{aligned} \kappa_{\mathcal{I}}(\phi) &= \log(\mathcal{I} - (\mathcal{I}-1)e^{\phi}) \\ &\quad - \log(\mathcal{I}(1-e^{\phi})). \end{aligned} \qquad (8)$$

Theorem 6 *The NEF from EFZMGL that corresponds to value $\mathcal{I} \in \mathbf{R}^1_+$ of invariant (7) admits the following canonical representation:*

$$\mathbf{P}^{(\phi)}(k) = e^{\phi \cdot k} \cdot e^{-\kappa_{\mathcal{I}}(\phi)} \cdot \nu_{\mathcal{I}}(\{k\}).$$

Here, $\phi \in \Phi$ and $k \in \mathbf{Z}_+$.

The u.v.f. $V_{\mathcal{I}}(\mu)$ of each such NEF admits the following closed-form representation:

$$V_{\mathcal{I}}(\mu) = \mu \cdot \sqrt{\mu^2 + (4\mathcal{I} - 2)\mu + 1}. \qquad (9)$$

Letac-Mora ([4]) self-reciprocity:

Theorem 7 $\forall \mathcal{I} \in \mathbf{R}^1_+$,

$$-\kappa_{\mathcal{I}}(-\kappa_{\mathcal{I}}(\phi)) \equiv \phi;$$

$$V_{\mathcal{I}}(\mu) \equiv \mu^3 \cdot V_{\mathcal{I}}(1/\mu).$$

It can be shown that all the members of EFZMGL for which $\mathcal{I} \geq 1/2$, are infinitely divisible. Hence, they can be emloyed to build the corresponding exponential families of Lévy processes. The remaining representatives of this family which correspond to the values of $\mathcal{I} \in [0, 1/2)$, are not infinitely divisible.

The next result is derived from the closed-form expression for the index of dispersion of a member of EFZMGL (which follows from (9)), and representation (7) for invariant $\mathcal{I}$:

Proposition 8 *Fix an arbitrary* $\gamma \in (0, 1)$. *Then r.v.* $Y_{\gamma,r}$ *is underdispersed if*
$r \in [-(1-\gamma)/\gamma, -(1-\gamma)/(2\gamma))$.

In the cases where $r = -(1-\gamma)/(2\gamma)$ *and* $r \in (-(1-\gamma)/(2\gamma), 1)$, *this r.v. is equidispersed and overdispersed, respectively.*

Hence, we found additional examples of under- and equidispersed r.v.'s. Member of EFZMGL can be either unimodal with mode at either 0 or 1 or bimodal with modes at 0 and 1.

References

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