

Lie groupoids for space robots

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Anecdotes from an interesting collaboration

with Robin Chhabra and Reza Emami

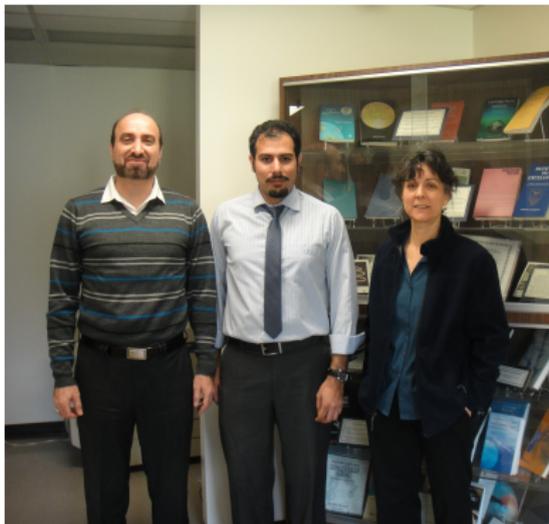
A UNIFIED GEOMETRIC FRAMEWORK FOR
KINEMATICS, DYNAMICS AND CONCURRENT CONTROL OF
FREE-BASE, OPEN-CHAIN MULTI-BODY SYSTEMS WITH
HOLOMOMIC AND NONHOLOMOMIC CONSTRAINTS

by

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A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Aerospace Science and Engineering
University of Toronto

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Graduate Texts in Mathematics

V.I. Arnold

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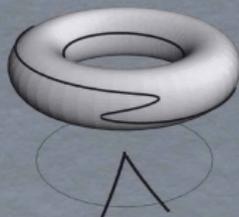
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GEOMETRIC FUNDAMENTALS OF ROBOTICS

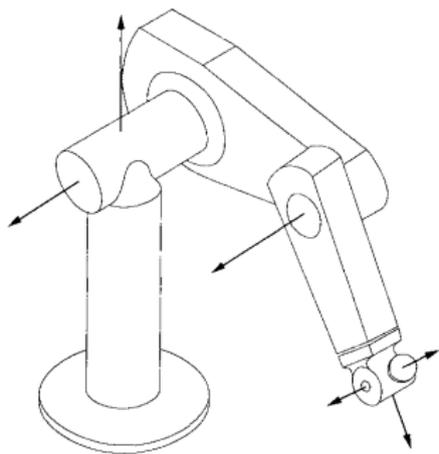
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Robot arm with six joints

Euclidean motions:

$$SE(3) = \{q: x \mapsto Ax + b\}$$

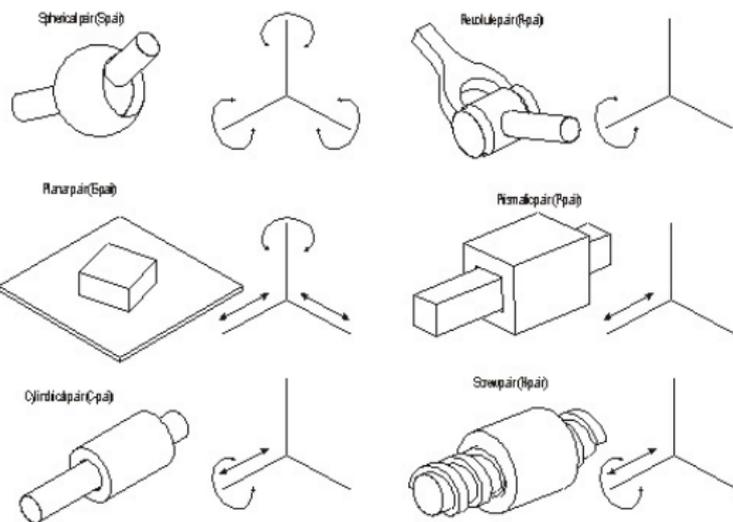
$$x \in \mathbb{R}^3, A \in SO(3), b \in \mathbb{R}^3.$$

$$1 \longrightarrow \mathbb{R}^3 \hookrightarrow \text{SE}(3) \xrightarrow{\pi} \text{SO}(3) \longrightarrow 1$$

$$q \in \mathbb{R}^3_{\text{affine}} \xrightarrow{\pi} \pi(q) \in \mathbb{R}^3_{\text{linear}}$$

“macro”: $\pi(q)$ is “ q viewed from far away”.

“micro”: $\forall x \ T_x \mathbb{R}^3_{\text{affine}} = \mathbb{R}^3_{\text{linear}}; \ \pi(q) = dq|_x$.



Lower Reuleaux pairs: spherical, planar, cylindrical, revolute, prismatic, screw

Connected subgroups G of $SE(3)$:

$$1 \longrightarrow \underbrace{K^c}_{=G \cap \mathbb{R}^3} \longrightarrow G \longrightarrow \underbrace{\pi(G)}_{=1 \text{ or } S^1 \text{ or } SO(3)} \longrightarrow 1$$

Table 2.1: Categories of displacement subgroups [38, 71]

| Dim. | Subgroups of $SE(3)$ /displacement subgroups | | | |
|------|--|--------------------|--------------------|---|
| 6 | $SE(3) = SO(3) \times \mathbb{R}^3$ free ^a | | | |
| 4 | $SE(2) \times \mathbb{R}$ planar+prismatic ^b | | | |
| 3 | $SE(2) = SO(2) \times \mathbb{R}^2$ | $SO(3)$ | \mathbb{R}^3 | $H_p \times \mathbb{R}^2$ |
| | planar | ball (spherical) | 3-d.o.f. prismatic | helical + 2-d.o.f. prismatic ^c |
| 2 | $SO(2) \times \mathbb{R}$ | \mathbb{R}^2 | | |
| | cylindrical ^d | 2-d.o.f. prismatic | | |
| 1 | $SO(2)$ | \mathbb{R} | H_p | |
| | revolute | prismatic | helical | |
| 0 | $\{e\}$ fixed ^a | | | |

^a These two subgroups are the trivial subgroups of $SE(3)$.

^b The axis of the prismatic joint is always perpendicular to the plane of the planar joint.

^c The axis of the helical joint is always perpendicular to the plane of the 2-d.o.f. prismatic joint.

^d The axis of the revolute and prismatic joints are always aligned.

One parameter subgroups

$$(\mathbb{R}, +) \longrightarrow \text{SE}(3)$$

THE
THEORY OF SCREWS:

*A STUDY IN THE DYNAMICS OF
A RIGID BODY.*

BY

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HODGES, FOSTER, AND CO., GRAFTON-STREET.
BOOKSELLERS TO THE UNIVERSITY.

1876.

$$\left\{ \begin{array}{l} \text{one-parameter subgroups} \\ (\mathbb{R}, +) \rightarrow \text{SE}(3) \end{array} \right\} = \left\{ \text{screws} \right\}$$

$$\mathfrak{se}(3) = \{\text{twists}\}$$

$$\mathfrak{se}(3)^* = \{\text{wrenches}\}$$

Chasles's Theorem:

Every Euclidean motion in 3-d is a screw motion.

Proof of Chasles's theorem.

$$q \in \text{SE}(3) \quad \mapsto \quad \pi(q) \in \text{SO}(3).$$

$$\pi(q) = \text{Id} \quad \Rightarrow \quad q \text{ is a translation.}$$

$$\pi(q) \neq \text{Id} \quad \xRightarrow{\text{Euler}} \quad \pi(q) \text{ is a rotation about line } \ell \subset \mathbb{R}_{\text{linear}}^3$$

$$\Rightarrow \quad q \text{ descends to } (\bar{q} \in \mathbb{R}_{\text{affine}}^3 / \ell) \in \text{SO}(2);$$

$$\bar{q} \text{ is a rotation about } \bar{x} = x + \ell \in \mathbb{R}_{\text{affine}}^3 / \ell$$

$$\Rightarrow \quad q \text{ is a screw motion about } x + \ell \subset \mathbb{R}_{\text{affine}}^3.$$

Dynamics.

$$\text{Lagrangian} = \underbrace{\text{Kinetic energy}}_{\sum_{\text{particles}} \frac{mv^2}{2}} - \underbrace{\text{Potential energy}}_{\text{assume } =0}$$

Configuration space: $Q = \{q\}$.

Velocity phase space: $TQ = \{(q, \dot{q})\}$, $\dot{q} \in T_q Q$.

Lagrangian: $TQ \rightarrow \mathbb{R}$.

Time evolution: $\{q_t\}_{a \leq t \leq b}$, path in Q .

 prolongation $\rightarrow \{(q_t, \dot{q}_t)\}_{a \leq t \leq b}$, path in TQ .

Principle of stationary action: $\delta \int_a^b L(q, \dot{q}) dt = 0$

\Rightarrow Euler-Lagrange equations

$$\frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}$$

Rigid body.

Configuration space: $Q \cong SE(3)$

infinitesimal motion: $\dot{q} \in T_e SE(3) = \mathfrak{se}(3)$,
a vector field on \mathbb{R}^3 :

$$\dot{q}|_x = \dot{x}.$$

$$\begin{aligned} \text{Kinetic energy} &= \frac{1}{2} \int_{x \in \text{body}} |\dot{x}|^2 \underbrace{d\rho(x)}_{\text{mass density}} \left(= \text{“} \sum_{\text{particles}} \frac{mv^2}{2} \text{”} \right) \\ &= K(\dot{q}, \dot{q}) \end{aligned}$$

$$\text{where } K(\dot{q}_1, \dot{q}_2) = \int_{x \in \text{body}} \langle \dot{q}_1|_x, \dot{q}_2|_x \rangle d\rho(x)$$

$Q \cong \text{SE}(3)$; Kinetic energy = $K(\dot{q}, \dot{q})$;

$K(\cdot, \cdot)$ an inner product on $\mathfrak{se}(3)$

~~~~~>  $6 \times 6$  “generalized inertia matrix”;

Lagrangian = norm-squared:  $TQ \rightarrow \mathbb{R}$

for left invariant Riemannian metric.

## Lie groupoid.

$G_0$  objects  
 $G_1$  arrows } manifolds

$s: G_1 \rightarrow G_0$  source map  
 $t: G_1 \rightarrow G_0$  target map } submersions

$h, g \mapsto h \cdot g$  multiplication on  $G_1$   
defined when  $t(g) = s(h)$  } associative

units:  $G_0 \rightarrow G_1, \quad a \mapsto 1_a$   
inverses:  $G_1 \rightarrow G_1, \quad g \mapsto g^{-1}$  } smooth

## Multibody system

Objects: the bodies  $B_1, \dots, B_N$ .

$A_i$  an affine space “attached to  $B_i$ ”.

Arrows from  $i$  to  $j$ :  $\{r_i^j: A_i \xrightarrow{\text{Euclidean}} A_j\}$

= { relative poses of  $B_i$  with respect to  $B_j$  }

“homing”  
 $\cong$  SE(3)