Lie groupoids for space robots

Yael Karshon

University of Toronto

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Anecdotes from an interesting collaboration

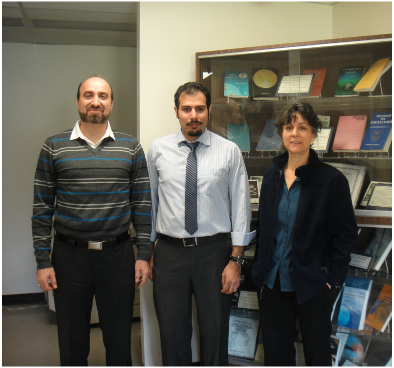
with Robin Chhabra and Reza Emami

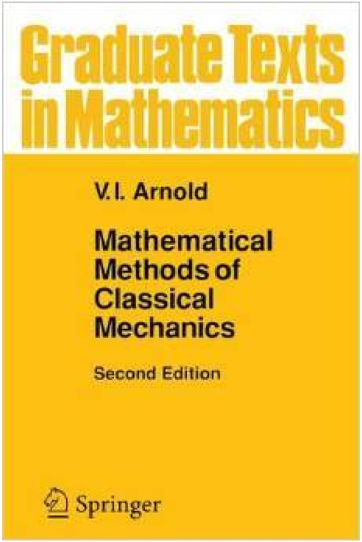
A Unified Geometric Framework for
Kinematics, Dynamics and Concurrent Control of
Free-base, Open-chain Multi-body Systems with
Holonomic and Nonholonomic Constraints

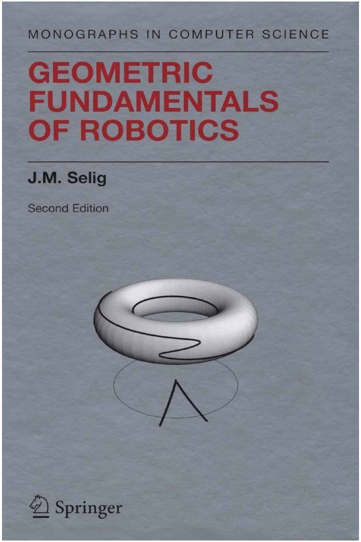
by

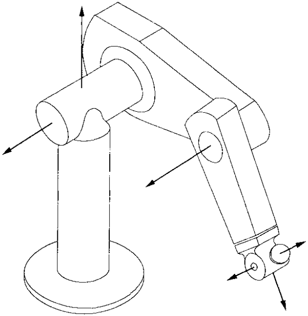
Robin Chhabra

A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Aerospace Science and Engineering
University of Toronto




Graduate Texts in Mathematics
V.I. Arnold
Mathematical Methods of Classical Mechanics
Second Edition
Springer


MONOGRAPHS IN COMPUTER SCIENCE
GEOMETRIC FUNDAMENTALS OF ROBOTICS
J.M. Selig
Second Edition
Springer



Robot arm with six joints

Euclidean motions:

$$\mathrm{SE}(3) = \{q\colon x \mapsto Ax + b\}$$

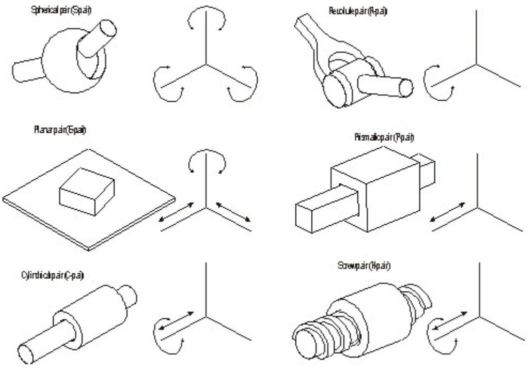
$x \in \mathbb{R}^3$, $A \in \mathrm{SO}(3)$, $b \in \mathbb{R}^3$.

$$1 \longrightarrow \mathbb{R}^3 \hookrightarrow \mathrm{SE}(3) \xrightarrow{\pi} \mathrm{SO}(3) \longrightarrow 1$$

$$q \circlearrowright \mathbb{R}^3_{\text{affine}} \quad \xmapsto{\pi} \quad \pi(q) \circlearrowright \mathbb{R}^3_{\text{linear}}$$

"macro": $\pi(q)$ is "q viewed from far away".

"micro": $\forall x \;\; T_x\mathbb{R}^3_{\text{affine}} = \mathbb{R}^3_{\text{linear}}; \quad \pi(q) = dq|_x$.



Lower Reuleaux pairs: spherical, planar, cylindrical, revolute, prismatic, screw

Connected subgroups G of SE(3):

$$1 \longrightarrow \underbrace{K}_{=G\cap\mathbb{R}^3} \hookrightarrow G \longrightarrow \underbrace{\pi(G)}_{=1 \text{ or } S^1 \text{ or } \mathrm{SO}(3)} \longrightarrow 1$$

Table 2.1: Categories of displacement subgroups [38, 71]

Dim.	Subgroups of $SE(3)$/displacement subgroups			
6	$SE(3) = SO(3) \ltimes \mathbb{R}^3$ free[a]			
4	$SE(2) \times \mathbb{R}$ planar+prismatic[b]			
3	$SE(2) = SO(2) \ltimes \mathbb{R}^2$ planar	$SO(3)$ ball (spherical)	$\mathbb{R}^3$ 3-d.o.f. prismatic	$H_p \ltimes \mathbb{R}^2$ helical + 2-d.o.f. prismatic[c]
2	$SO(2) \times \mathbb{R}$ cylindrical[d]	$\mathbb{R}^2$ 2-d.o.f. prismatic		
1	$SO(2)$ revolute	$\mathbb{R}$ prismatic	H_p helical	
0	$\{e\}$ fixed[a]			

[a] These two subgroups are the trivial subgroups of $SE(3)$.
[b] The axis of the prismatic joint is always perpendicular to the plane of the planar joint.
[c] The axis of the helical joint is always perpendicular to the plane of the 2-d.o.f. prismatic joint.
[d] The axis of the revolute and prismatic joints are always aligned.

One parameter subgroups

$$(\mathbb{R}, +) \longrightarrow \mathsf{SE}(3)$$

THE

THEORY OF SCREWS:

A STUDY IN THE DYNAMICS OF A RIGID BODY.

BY

ROBERT STAWELL BALL, LL.D., F.R.S.,

ANDREWS' PROFESSOR OF ASTRONOMY IN THE UNIVERSITY OF DUBLIN,
AND ROYAL ASTRONOMER OF IRELAND.

DUBLIN:
HODGES, FOSTER, AND CO., GRAFTON-STREET.
BOOKSELLERS TO THE UNIVERSITY.
1876.

$$\left\{\begin{matrix}\text{one-parameter subgroups} \\ (\mathbb{R}, +) \to \mathsf{SE}(3)\end{matrix}\right\} = \left\{\text{screws}\right\}$$

$$\mathfrak{se}(3) = \{\text{twists}\}$$
$$\mathfrak{se}(3)^* = \{\text{wrenches}\}$$

Chasles's Theorem:

Every Euclidean motion in 3-d is a screw motion.

Proof of Chasles's theorem.

$$q \in \mathrm{SE}(3) \qquad \mapsto \qquad \pi(q) \in \mathrm{SO}(3).$$

$$\pi(q) = \mathrm{Id} \quad \Rightarrow \quad q \text{ is a translation.}$$

$$\pi(q) \neq \mathrm{Id} \quad \overset{\text{Euler}}{\Rightarrow} \quad \pi(q) \text{ is a rotation about line } \ell \subset \mathbb{R}^3_{\text{linear}}$$

$$\Rightarrow \quad q \text{ descends to } \left(\overline{q} \circlearrowright \mathbb{R}^3_{\text{affine}}/\ell\right) \in \mathrm{SO}(2);$$

$$\overline{q} \text{ is a rotation about } \overline{x} = x + \ell \in \mathbb{R}^3_{\text{affine}}/\ell$$

$$\Rightarrow q \text{ is a screw motion about } x + \ell \subset \mathbb{R}^3_{\text{affine}}.$$

Dynamics.

$$\text{Lagrangian} = \underbrace{\text{Kinetic energy}}_{\sum_{\text{particles}} \frac{mv^2}{2}} - \underbrace{\text{Potential energy}}_{\text{assume } =0}$$

Configuration space: $Q = \{q\}$.
Velocity phase space: $TQ = \{(q, \dot{q})\}, \quad \dot{q} \in T_q Q$.

$$\text{Lagrangian}\colon TQ \to \mathbb{R}.$$

Time evolution: $\{q_t\}_{a\le t\le b}$, path in Q.

$\underset{\text{prolongation}}{\rightsquigarrow} \{(q_t, \dot{q}_t)\}_{a\le t\le b}$, path in TQ.

Principle of stationary action: $\delta \int_a^b L(q,\dot{q})dt = 0$

$\Rightarrow$ Euler-Lagrange equations

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

Rigid body.

Configuration space: $Q \cong \mathrm{SE}(3)$

infinitesimal motion: $\dot{q} \in T_e SE(3) = \mathfrak{se}(3)$,
a vector field on $\mathbb{R}^3$:

$$\dot{q}|_x = \dot{x}.$$

$$\begin{aligned}\text{Kinetic energy } &= \frac{1}{2} \int_{x \in \text{body}} |\dot{x}|^2 \underbrace{d\rho(x)}_{\text{mass density}} \left(= \text{“} \sum_{\text{particles}} \frac{mv^2}{2} \text{”} \right) \\ &= K(\dot{q}, \dot{q})\end{aligned}$$

where $K(\dot{q}_1, \dot{q}_2) = \int_{x \in \text{body}} \langle \dot{q}_1|_x, \dot{q}_2|_x \rangle d\rho(x)$

$Q \cong \mathrm{SE}(3)$; Kinetic energy $= K(\dot{q}, \dot{q})$;

$K(\cdot, \cdot)$ an inner product on $\mathfrak{se}(3)$

$\rightsquigarrow$ 6×6 "generalized inertia matrix";

$$\text{Lagrangian} = \text{norm-squared}\colon\ TQ \to \mathbb{R}$$

for left invariant Riemannian metric.

Lie groupoid.

$$\left.\begin{array}{l} G_0 \text{ objects} \\ G_1 \text{ arrows} \end{array}\right\} \text{manifolds}$$

$$\left.\begin{array}{ll} s\colon G_1 \to G_0 & \text{source map} \\ t\colon G_1 \to G_0 & \text{target map} \end{array}\right\} \text{submersions}$$

$$\left.\begin{array}{l} h, g \mapsto h \cdot g \text{ multiplication on } G_1 \\ \text{defined when } t(g) = s(h) \end{array}\right\} \text{associative}$$

$$\left.\begin{array}{l} \text{units: } G_0 \to G_1, \quad a \mapsto 1_a \\ \text{inverses: } G_1 \to G_1, \quad g \mapsto g^{-1} \end{array}\right\} \text{smooth}$$

Multibody system

Objects: the bodies $B_1, \dots, B_N$.

A_i an affine space "attached to B_i".

Arrows from i to j: $\{ r_i^j : A_i \xrightarrow[\text{Euclidean}]{} A_j \}$

$= \{$ relative poses of B_i with respect to B_j $\}$

$\overset{\text{"homing"}}{\cong}$ SE(3)