Local invariants of maps between 3-manifolds

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History

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Arnold	semi-local invariants of order 1 of plane curves and fronts

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VG, Houston Nowik	local order 1 invariants of maps of surfaces to $\mathbb{R}^3$
Ohmoto Aicardi	local order 1 invariants of maps of surfaces to $\mathbb{R}^2$

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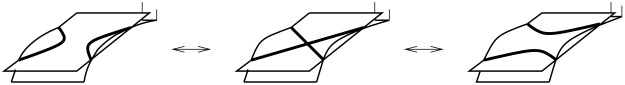

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Further details and other orientation settings in

VG, *Local invariants of maps between 3-manifolds,* Journal of Topology **6** (2013) 757-776

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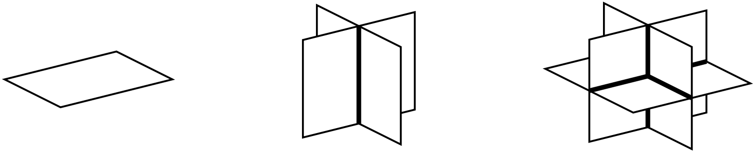

A_1 A_1^2 A_1^3

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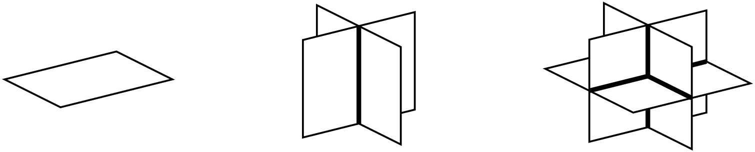

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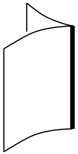
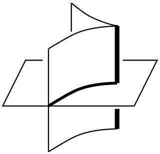
Co-orientation of the regular part of $\mathcal{C}$:

towards its side with more local preimages



A_2



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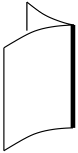

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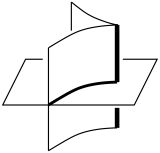

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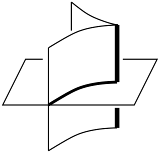
Cuspidal edges: positive and negative
according to the local degree of the map being ± 1



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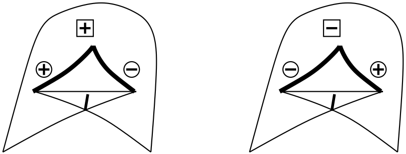
Hence signs for swallowtails:



A_3^+ A_3^-

Examples of local invariants

6 obvious:

I_t, the number of triple points A_1^3;

$I_{s_\pm}$, the numbers of positive and negative swallowtails;

$I_{c_\pm}$, the numbers of $A_2^\pm A_1$ points;

I_χ, the Euler characteristic of the critical locus $\mathcal{K} \subset M$.

Linking invariant $I_{\sum^2}$

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Orient φ as $[0,1] \times M$. Due to the parallelisability of M and N, we have well-defined value

$$I_{\Sigma^2}(f_1) = \langle \varphi, \Sigma^2 \rangle + I_{\Sigma^2}(f_0)$$

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$$(I_{s_+} \pm I_{s_-})/2\,, \quad (I_{c_+} + I_{c_-})/2\,, \quad I_t\,, \quad (I_t + I_{c_+})/2\,, \quad I_\chi/2\,, \quad I_{\Sigma^2}\,.$$

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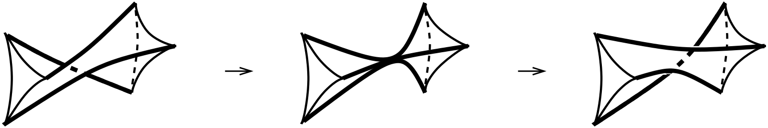

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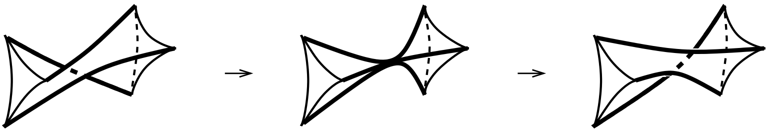
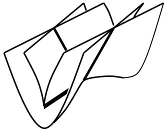


By this transition we co-orient the D_4^{-+} stratum.
The co-orientation of D_4^{--} is in the opposite direction.
Both co-orientations correspond to the increase of the deformation parameter λ.

$D_4^{+\pm}: (x^2 + y^2 + zy + \lambda x, \pm xy, z)$, where $\pm$ is the edge sign for $\lambda = 0$:



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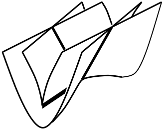

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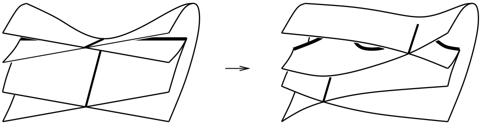

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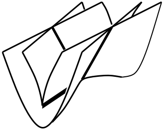

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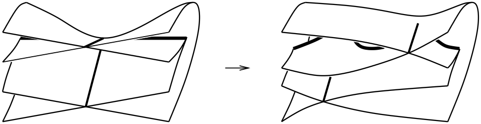
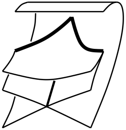


Half of the right surface:



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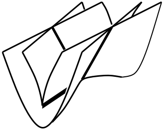

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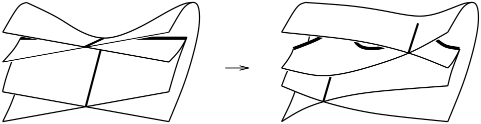

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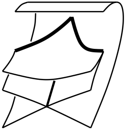
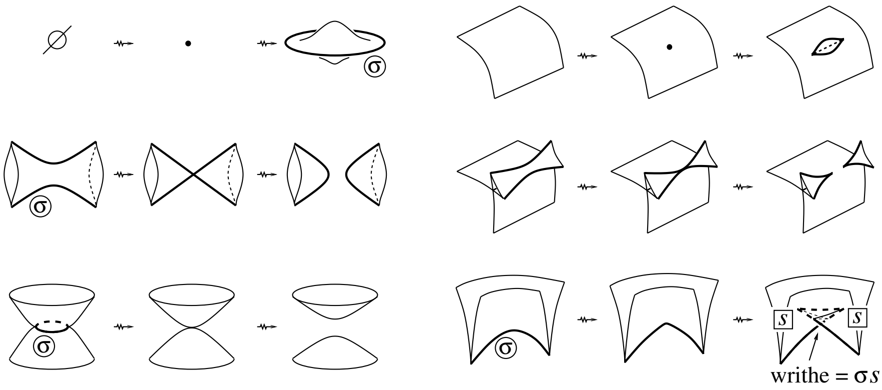


Co-orientation:

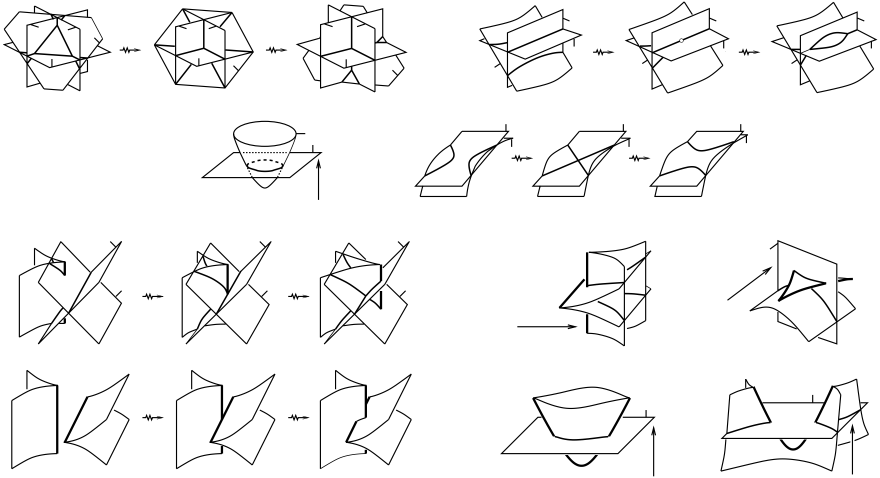
by the sign of the swallowtails, equivalently by the increase of λ

Catalog of 1-parameter bifurcations of cork 1 maps

Uni-germs



Multi-germs



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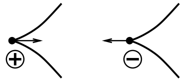
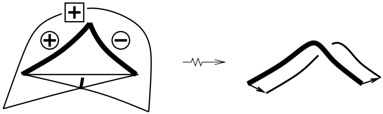
The $I_{L_\pm}$ invariants are due to Franka Aicardi.

Framed link from the cuspidal edge





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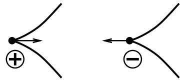

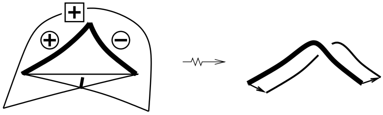

Orient arbitrarily the framed link.

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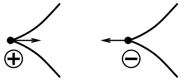

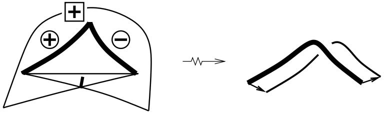

Orient arbitrarily the framed link.

Let w be its *writhe*, that is, the algebraic number of crossings of the cores of the components in its link diagram

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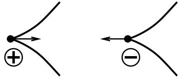
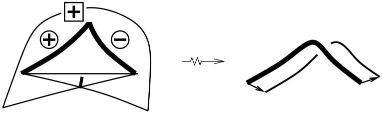
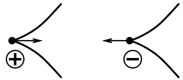


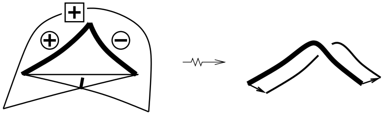


Orient arbitrarily the framed link.

Let w be its *writhe*, that is, the algebraic number of crossings of the cores of the components in its link diagram plus the sum of the algebraic numbers of full rotations done by the framing of each of the components around its own core.

Framed link from the cuspidal edge





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Since the number of crossings of two different components is even, w mod4 does not depend on the orientations of the components.

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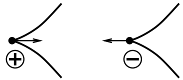

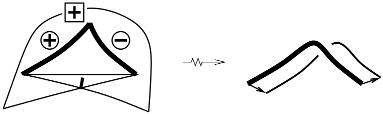

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Let n be the number of components of the link.

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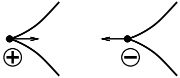

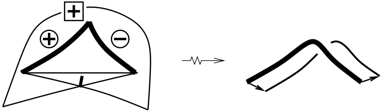

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Since the number of crossings of two different components is even, $w \bmod 4$ does not depend on the orientations of the components.
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Theorem The mod2 invariant $I_{fe} = n + w/2$ is local.

Lemma

Consider two local modifications of a framed link:

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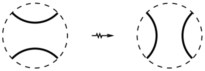

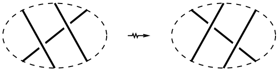

Assume the framing of all participating fragments is blackboard.

Lemma

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Assume the framing of all participating fragments is blackboard. Then the 1st move changes $(n + w/2)\,\mathrm{mod}2$ by 1, while the 2nd preserves this number.

Framed link L_+ from the positive edges and selfintersection

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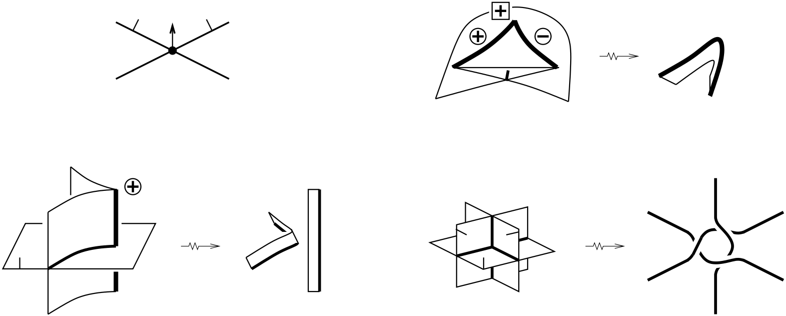

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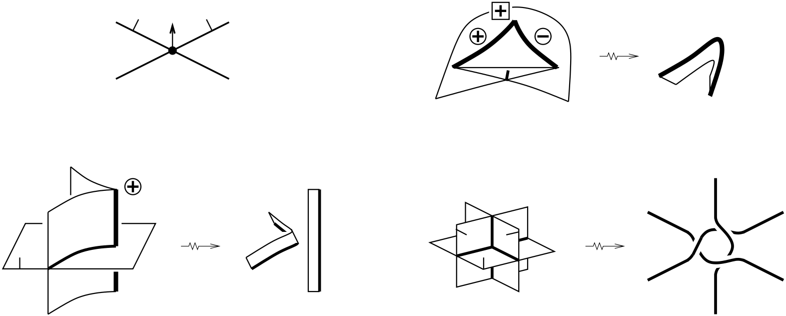

The invariant I_{L_+} is similar to I_{fe} plus half the number of triple points

Corollary of the last Theorem

The rank of the mod2 invariant space for maps between two oriented 3-manifolds is at least 7 and at most 11.

Non-oriented source

The setting eliminates the signs of edges and swallowtails.

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$I_s/2$, half of the total number of swallowtails of the critical value set $\mathcal{C}$,

$I_c/2$, half of the number of A_2A_1 points of $\mathcal{C}$,

I_t, the number of triple points of $\mathcal{C}$, and

$I_\chi/2$, half of the Euler characteristic of the critical locus.

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- $I_\chi/2$, half of the Euler characteristic of the critical locus.

Reason: the claim holds for $\mathbb{R}^3$ as the target, since integer I_{Σ^2} requires orientation of the source.

Theorem

a) The space of the mod2 invariants of maps from a non-orientable 3-manifold to $\mathbb{R}^3$ has rank 6.

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Theorem

a) The space of the mod2 invariants of maps from a non-orientable 3-manifold to $\mathbb{R}^3$ has rank 6. Its basis is formed by $I_s/2,\ I_c/2,\ I_t,\ I_\chi/2,\ I_{\Sigma^2}$ and $I_{\Sigma^{1,1,1,1}}$.

b) If the target is arbitrary, then the rank of the mod2 invariant space is at least 4 and at most 6.

Oriented source and non-oriented target

I_{Σ^2} survives over $\mathbb{Z}$ for $\mathbb{R}^3$ as the target.

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Hence the space of integer invariants of maps to $\mathbb{R}^3$ has rank 5.
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The mod2 statement is the same as for a non-oriented source.