Algebro-geometric approach to the Schlesinger equations

with V. Shramchenko

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Legacy of V. I. Arnold, Fields Institute, Toronto, November 25, 2014

The title could be "On a solution of a differential equation..." as suggested by V. I. Arnold.

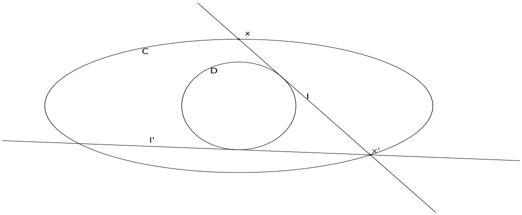
Six Painlevé equations

- Paul Painlevé (1863-1933) classified all second order ODEs of the form $\frac{d^2y}{dx^2} = F(\frac{dy}{dx}, y, x)$ with F rational in the first two arguments whose solutions have no movable singularities.
- Six new equations which cannot be solved in terms of known special functions.
- The sixth Painlevé equation, PVI, is the most general of them: PVI$(\alpha, \beta, \gamma, \delta)$.

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx}$$

$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right).$$

Poncelet problem



- C and D are two smooth conics in $\mathbb{CP}^2$
- Question: Is there a closed trajectory inscribed in C and circumscribed about D?
- Poncelet Theorem: Let $x \in C$ be a starting point. The Poncelet trajectory originating at x closes up after n steps iff so does a Poncelet trajectory originating at any other point of C.

Solution of Poncelet problem

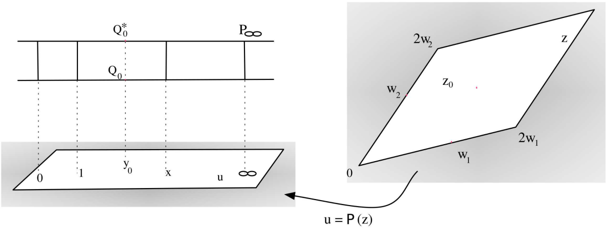
Griffiths, P., Harris, J., On Cayley's explicit solution to Poncelet's porism (1978)

- Let C and D be symmetric 3×3 matrices defining the conics C and D in $\mathbb{CP}^2$.
- $E = \{(x, y) \in \mathbb{CP}^1 \times \mathbb{CP}^1 : x \in C, y \in D^*, x \in y\}$ is an elliptic curve of the equation $v^2 = \det(D + uC)$.
- A closed Poncelet trajectory of length k exists for two conics C and D iff the point $(u, v) = (0, \sqrt{\det D})$ is of order k on E.
- $k\mathcal{A}_\infty(Q_0) \equiv 0 <=> \exists f \in L(-kP_\infty)$ with zero of order k at Q_0.

Hitchin's work

Hitchin, N. Poncelet polygons and the Painlevé equations (1992)

For two conics and a Poncelet trajectory of length k there is an associated algebraic solution of $PVI(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$.



- Existence of the Poncelet trajectory of length k implies $kz_0 \equiv 0$. $\quad (z_0 := 2w_1\frac{m_1}{k} + 2w_2\frac{m_2}{k}.)$
- $z_0 = \mathcal{A}_\infty(Q_0)$, where $\mathcal{A}_\infty$ is the Abel map based at P_∞.
- A function $g(u, v)$ on the curve $v^2 = u(u-1)(u-x)$ having a zero of order k at Q_0 and a pole of order k at P_∞

Hitchin's work

Hitchin, N. Poncelet polygons and the Painlevé equations (1992)

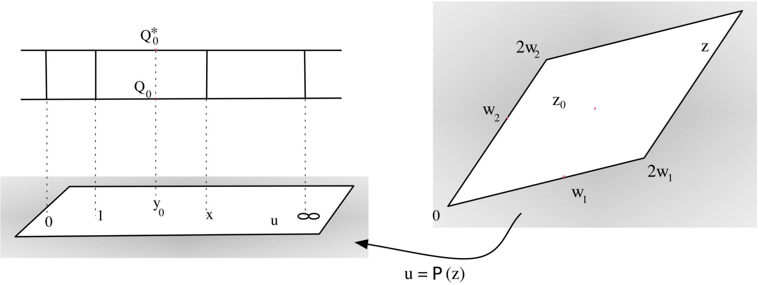
- The function
$$s(u,v) = \frac{g(u,v)}{g(u,-v)}$$
has a zero of order k at Q_0 and a pole of order k at Q_0^* and no other zeros or poles.

- ds has exactly two zeros away from Q_0 and Q_0^*.

- These two zeros are paired by the elliptic involution.

- Their u-coordinate as a function of x solves $PVI(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$.

Picard solution to PVI $(0, 0, 0, \frac{1}{2})$



- Transformed $\wp$ satisfies:
  $(\wp'(z))^2 = \wp(z)\,(\wp(z) - 1)\,(\wp(z) - x)$.
- Define

$$z_0 := 2w_1c_1 + 2w_2c_2.$$

- $z_0 = \mathcal{A}_\infty(Q_0)$.
- Picard's solution to PVI $(0, 0, 0, \frac{1}{2})$:

$$y_0(x) = \wp(z_0(x)).$$

Hitchin's solution of PVI$(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$

Twistor spaces, Einstein metrics and isomonodromic deformations (1995)

$$y(x) = \frac{\theta_1'''(0)}{3\pi^2\theta_4^4(0)\theta_1'(0)} + \frac{1}{3}\left(1 + \frac{\theta_3^4(0)}{\theta_4^4(0)}\right)$$
$$+ \frac{\theta_1'''(\nu)\theta_1(\nu) - 2\theta_1''(\nu)\theta_1'(\nu) + 4\pi \mathrm{i} c_2[\theta_1''(\nu)\theta(\nu) - \theta_1'^2(\nu)]}{2\pi^2\theta_4^4(0)\theta_1(\nu)[\theta_1'(\nu) + 2\pi \mathrm{i} c_2\theta_1(\nu)]}.$$

- Here $\nu = c_2\tau + c_1$ with $\tau = \frac{w_2}{w_1}$; and

$$x = \frac{\theta_3^4(0)}{\theta_4^4(0)}.$$

Okamoto transformations $\sim$ 1980

- a group of symmetries of PVI$(\alpha, \beta, \gamma, \delta)$.

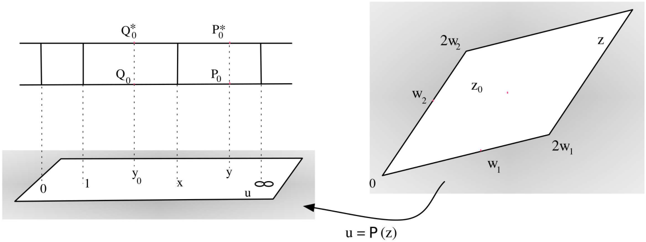
- Lemma (V. D., V. Shramchenko): Okamoto transformation from PVI$(0, 0, 0, \frac{1}{2})$ to PVI$(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$:

y_0 - Picard's solution

y - Hitchin's solution

$$y(x) = y_0 + \frac{y_0(y_0 - 1)(y_0 - x)}{x(x-1)y_0' - y_0(y_0 - 1)}.$$

Our construction



- $z_0 = 2w_1c_1 + 2w_2c_2, \quad z_0 = \mathcal{A}_\infty(Q_0), \quad y_0(x) = \wp(z_0(x)).$
- Differential of the third kind on the elliptic curve $\mathcal{C}$:

$$\Omega(P) = \Omega_{Q_0,Q_0^*}(P) - 4\pi \mathrm{i} c_2 \omega(P).$$

- $\omega(P)$ -holomorphic normalized differential on $\mathcal{C}$ in terms of z has the form: $\omega = \frac{dz}{2w_1}$.
- Ω has two simple poles at Q_0 et Q_0^* which project to y_0, Picard's solution of PVI $(0, 0, 0, \frac{1}{2})$.
- Ω has two simple zeros at P_0 et P_0^* which project to y, Hitchin's solution of PVI$(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$.

Ω_{Q_0,Q_0*} as the Okamoto transformation

- Write the differential Ω in terms of the coordinate u:

$$\Omega(P) = \frac{\omega(P)}{\omega(Q_0)}\left[\frac{1}{u(P)-y_0} - \frac{I}{2w_1}\right] - 4\pi \mathrm{i} c_2 \omega(P).$$

where $I = \oint_a \frac{du}{(u-y_0)\sqrt{u(u-1)(u-x)}}$.
$y = u(P)$ is projection of zeros of Ω iff

$$\frac{1}{y-y_0} = \frac{I}{2w_1} + 4\pi \mathrm{i} c_2 \omega(Q_0).$$

- By differentiating the relation $\int_{P_\infty}^{Q_0} \omega = c_1 + c_2\tau$ with respect to x we find the derivative $\frac{dy_0}{dx}$:

$$\begin{aligned}\frac{dy_0}{dx} &= -\frac{1}{4}\Omega(P_x)\frac{\omega(P_x)}{\omega(Q_0)} \\ &= \frac{1}{4}\frac{\omega^2(P_x)}{\omega^2(Q_0)}\left[4\pi \mathrm{i}\, c_2\, \omega(Q_0) - \frac{1}{x-y_0} + \frac{I}{2w_1}\right].\end{aligned}$$

$\Omega_{Q_0,Q_{0*}}$ as the Okamoto transformation

- Thus we get for the relationship between y and y_0 :

$$\frac{1}{y-y_0} = 4\frac{\omega^2(Q_0)}{\omega^2(P_x)}\frac{dy_0}{dx} + \frac{1}{x-y_0}.$$

- The holomorphic normalized differential in terms of the u-coordinate has the form

$$\omega(P) = \frac{du}{2w_1\sqrt{u(u-1)(u-x)}}.$$

- Therefore

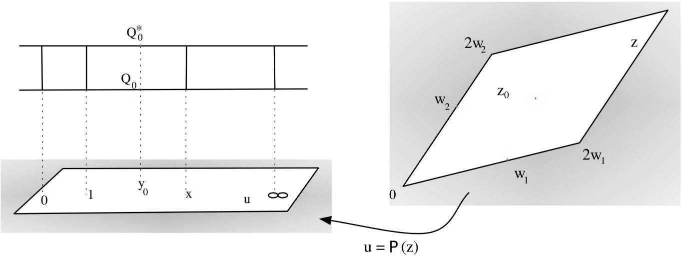
$$\omega(P_x) = \frac{2}{2w_1\sqrt{x(x-1)}} \quad \text{and} \quad \omega(Q_0) = \frac{1}{2w_1\sqrt{y_0(y_0-1)(y_0-x)}}.$$

- Okamoto transformation:

$$y(x) = y_0 + \frac{y_0(y_0-1)(y_0-x)}{x(x-1)y_0' - y_0(y_0-1)}.$$

Remark on $\frac{dy_0}{dx}$

$y_0(x) = \wp(z_0(x))$ - the Picard solution to PVI $(0, 0, 0, \frac{1}{2})$



$$\frac{dy_0}{dx} = -\frac{1}{4}\Omega(P_x)\frac{\omega(P_x)}{\omega(Q_0)}$$

$$\left(z_0 = 2w_1c_1 + 2w_2c_2 \qquad \Omega(P) = \Omega_{Q_0,Q_0*}(P) - 4\pi \mathrm{i} c_2\omega(P)\right)$$

Normalization of the differential Ω

- $z_0 = 2w_1c_1 + 2w_2c_2$.
- $\Omega(P) = \Omega_{Q_0,Q_0*}(P) - 4\pi \mathrm{i} c_2 \omega(P)$.
- The constants c_1 and c_2 determine the periods of Ω :

$$\oint_a \Omega = -4\pi \mathrm{i} c_2 \qquad \oint_b \Omega = 4\pi \mathrm{i} c_1.$$

- Ω does not depend on the choice of *a*- and *b*-cycles.
- Therefore our construction is global on the space of elliptic two-fold coverings of $\mathbb{C}P^1$ ramified above the point at infinity.

Schlesinger system (four points)

- Linear matrix system

$$\frac{d\Phi}{du} = A(u)\Phi, \qquad A(u) = \frac{A^{(1)}}{u} + \frac{A^{(2)}}{u-1} + \frac{A^{(3)}}{u-x}$$

$u \in \mathbb{C}, \Phi \in \mathrm{M}(2, \mathbb{C}), A \in sl(2, \mathbb{C})$

- Isomonodromy condition (Schlesinger system)

$$\frac{dA^{(1)}}{dx} = \frac{[A^{(3)}, A^{(1)}]}{x};$$
$$\frac{dA^{(2)}}{dx} = \frac{[A^{(3)}, A^{(2)}]}{x-1};$$
$$\frac{dA^{(3)}}{dx} = -\frac{[A^{(3)}, A^{(1)}]}{x} - \frac{[A^{(3)}, A^{(2)}]}{x-1}.$$

$$A^{(1)} + A^{(2)} + A^{(3)} = const.$$

Solution to the Schlesinger system (four points)

- By conjugating, assume $A^{(1)} + A^{(2)} + A^{(3)} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$.
- Then the term A_{12} is of the form:

$$A_{12}(u) = \kappa \frac{(u-y)}{u(u-1)(u-x)}$$

- The zero y as a function of x satisfies the

$$\mathrm{PVI}\left(\frac{(2\lambda-1)^2}{2},\ -\mathrm{tr}(A^{(1)})^2,\ \mathrm{tr}(A^{(2)})^2,\ \frac{1-2\mathrm{tr}(A^{(3)})^2}{2} \right)$$

- For PVI$(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$ $\lambda = -1/4$. Our construction implies

$$A_{12}(u) = \frac{\Omega(P)}{\omega(P)} \frac{(u-y_0)}{u(u-1)(u-x)}, \qquad P \in \mathcal{L},\ \ u = u(P).$$

Solution to the Schlesinger system (four points)

- Let $\phi(P) = \frac{du}{\sqrt{u(u-1)(u-x)}}$ - a non-normalized holom. diff.

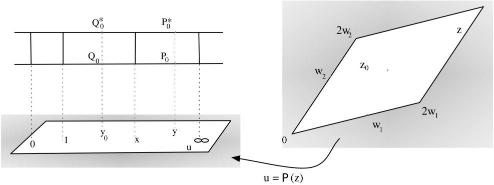
$$A_{12}^{(1)} = -\frac{1}{4} y_0 \Omega(P_0)\phi(P_0), \qquad \beta_1 := -\frac{y_0}{4}\left(\Omega(P_0)\right)^2,$$
$$A_{12}^{(2)} = \frac{1}{4}(1-y_0)\Omega(P_1)\phi(P_1), \qquad \beta_2 := \frac{1-y_0}{4}\left(\Omega(P_1)\right)^2,$$
$$A_{12}^{(3)} = \frac{1}{4}(x-y_0)\Omega(P_x)\phi(P_x), \qquad \beta_3 := \frac{x-y_0}{4}\left(\Omega(P_x)\right)^2.$$

- Then the following matrices solve the Schlesinger system

$$A^{(i)} := \begin{pmatrix} -\frac{1}{4} - \frac{\beta_i}{2} & A_{12}^{(i)} \\ -\frac{1}{4}\frac{\beta_i + \beta_i^2}{A_{12}^{(i)}} & \frac{1}{4} + \frac{\beta_i}{2} \end{pmatrix}, \qquad i = 1, 2, 3.$$

- Eigenvalues of matrices $A^{(i)}$ are $\pm 1/4$.
- cf. Kitaev, A., Korotkin, D. (1998); Deift, P., Its, A., Kapaev, A., Zhou, X. (1999)

Generalization to hyperelliptic curves



Let $z_0 \in Jac(\mathcal{L})$, $z_0 = c_1 + c_2^t \mathbb{B}$, and $\sum_{j=1}^{g} \mathcal{A}_\infty(Q_j) = z_0$.
Define the differential

$$\Omega(P) = \sum_{j=1}^{g} \Omega_{Q_j Q_j^*}(P) - 4\pi\mathrm{i}\, c_2^t \omega(P).$$

Let $q_j = u(Q_j)$. Then

$$\frac{\partial q_j}{\partial u_k} = -\frac{1}{4}\Omega(P_k) v_j(P_k),$$

where

$$v_j(P) = \frac{\phi(P)\prod_{\alpha=1,\alpha\neq j}^{g}(u - q_\alpha)}{\phi(Q_j)\prod_{\alpha=1,\alpha\neq j}^{g}(q_j - q_\alpha)}, \qquad j = 1, \ldots, g$$

Normalization of the differential Ω

$$\Omega(P) = \sum_{j=1}^{g} \Omega_{Q_j Q_j^\tau}(P) - 4\pi\mathrm{i}\, c_2^t \omega(P)$$

where $z_0 = c_1 + c_2^t \mathbb{B}$ and $\sum_{j=1}^{g} \mathcal{A}_\infty(Q_j) = z_0$;

$c_1, c_2 \in \mathbb{R}^g$.

- The constant vectors $c_1 = (c_{11}, \dots c_{1g})^t$ and $c_2 = (c_{21}, \dots, c_{2g})^t$ determine the periods of Ω :

$$\oint_{a_k} \Omega = -4\pi\mathrm{i} c_{2k} \qquad \oint_{b_k} \Omega = 4\pi\mathrm{i} c_{1k}.$$

- Ω does not depend on the choice of *a*- and *b*-cycles.

Schlesinger system (n points)

$$\frac{d\Phi}{du} = A(u)\Phi, \qquad A(u) = \sum_{j=1}^{2g+1} \frac{A^{(j)}}{u - u_j},$$

where $u \in \mathbb{C}$, $\Phi(u) \in \mathrm{M}(2,\mathbb{C})$, $A^{(j)} \in sl(2,\mathbb{C})$.

- Schlesinger system for residue-matrices $A^{(i)} \in sl(2,\mathbb{C})$:

$$\frac{\partial A^{(j)}}{\partial u_k} = \frac{[A^{(k)}, A^{(j)}]}{u_k - u_j}; \qquad A^{(1)} + \cdots + A^{(2g+1)} = -A^{(\infty)} = const$$

- by removing the conjugation freedom assume

$$A^{(\infty)} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}.$$

Solution to the Schlesinger system (n points)

- Let $\phi(P) = \frac{du}{\sqrt{\prod_{i=1}^{2g+1}(u-u_i)}}$ - a non-normalized holom. diff.
- Use the differential Ω to construct an analogue of A_{12} in the hyperelliptic case

$$A_{12}(u) = \frac{\Omega(P)}{\phi(P)} \frac{\prod_{\alpha=1}^{g}(u-q_\alpha)}{\prod_{j=1}^{2g+1}(u-u_j)},$$

- Its residues at the simple poles:

$$A_{12}^{(n)} = \frac{\kappa}{4}\Omega(P_n)\phi(P_n)\prod_{\alpha=1}^{g}(u_n - q_\alpha). \qquad (1)$$

- Introduce the following quantities:

$$\beta_n := \frac{1}{4}\Omega(P_n)\sum_{j=1}^{g} v_j(P_n) - \frac{1}{2}\Omega(\infty)A_{12}^{(n)}.$$

- The following matrices $A^{(i)}$ with $i = 1, \ldots, 2g+1$ solve the Schlesinger system

$$A^{(i)} := \begin{pmatrix} -\frac{1}{4} - \frac{\beta_i}{2} & A_{12}^{(i)} \\ -\frac{1}{4}\,\frac{\beta_i + \beta_i^2}{A_{12}^{(i)}} & \frac{1}{4} + \frac{\beta_i}{2} \end{pmatrix};$$

- $$A^{(1)} + \cdots + A^{(2g+1)} = -A^{(\infty)} = \begin{pmatrix} -1/4 & 0 \\ 0 & 1/4 \end{pmatrix}.$$
- cf. Kitaev, A., Korotkin, D. (1998); Deift, P., Its, A., Kapaev, A., Zhou, X. (1999)
- Zeros of Ω are zeros of $A_{12}(u)$ and are solutions of the multidimensional Garnier system.

Back to Poncelet

$$n = 2g + 2$$

Consider the case of a point z_0 with rational coordinates $c_1, c_2 \in \mathbb{Q}^g$ with respect to the Jacobian of the hyperelliptic curve of genus g. It corresponds to a periodic trajectory of a billiard ordered game associated to g quadrics from a confocal family in $d = g + 1$ dimensional space.
For billiard ordered games see V. Dragović, M. Radnović, JMPA 2006.