Steiner-Star-Free Graphs and Equivalence of Steiner Tree Relaxations

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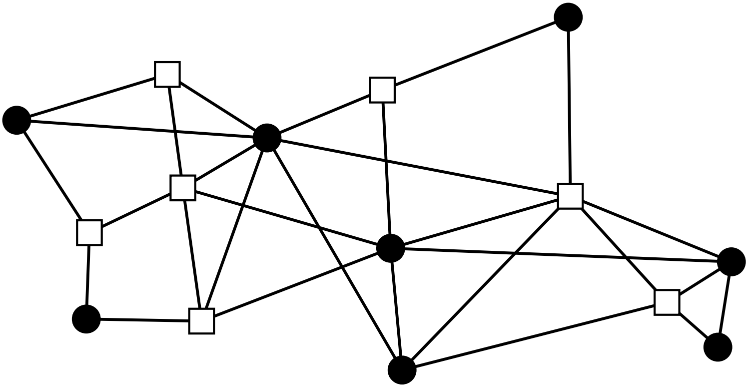
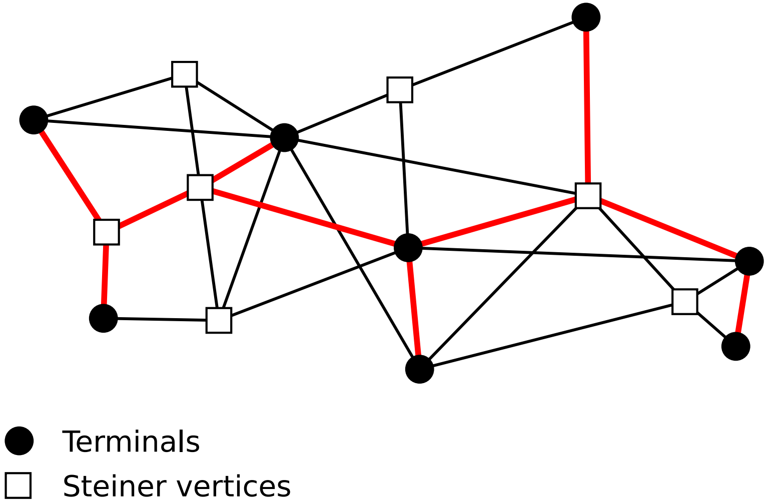
The Steiner Tree problem



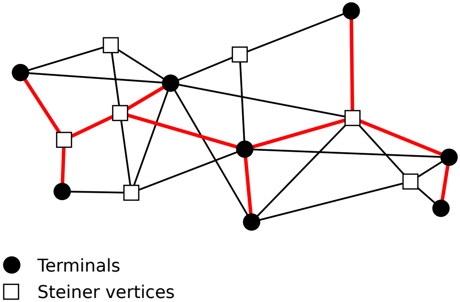
● Terminals

□ Steiner vertices

The Steiner Tree problem



Applications and known results



- applications:
  network design, VLSI
- one of Karp's original
  21 NP-hard problems
- APX-hard

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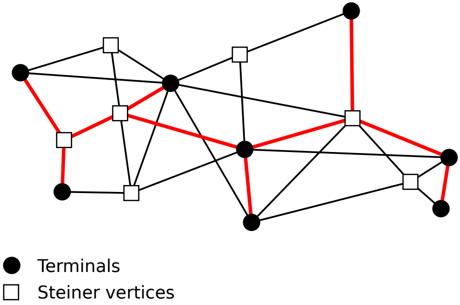

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- APX-hard

- $(\ln(4) + \varepsilon)$-approximation: [Byrka et al. 2012]
  - iterative rounding of *hypergraphic* (HYP) LP
  - solving HYP is strongly NP-hard
  - runtime bottleneck: PTAS for HYP

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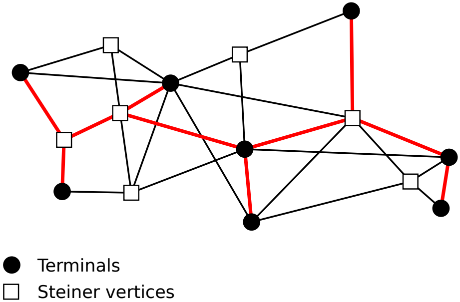

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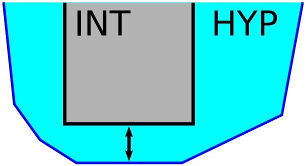
1. aim: improve runtime

Integrality gaps



- *hypergraphic* (HYP) LP:

  [Warme 1998]

  – strongly NP-hard to solve
  $\rightarrow$ PTAS necessary

  + HYP gap $\leq \ln(4) \approx 1.39$

    [Goemans et al. 2012]

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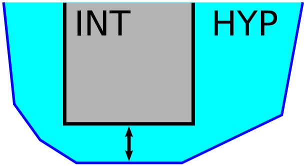

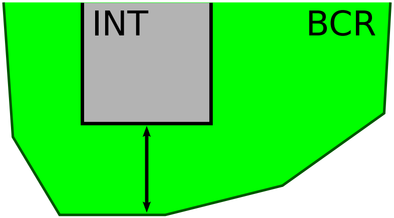

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- *bidirected cut* (BCR) LP:
  [Edmonds 1967]
  - + compact formulation
    $\rightarrow$ efficiently solvable
  - – BCR gap ≤ 2
    [Folklore]

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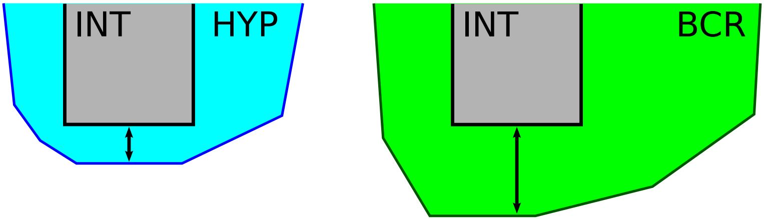

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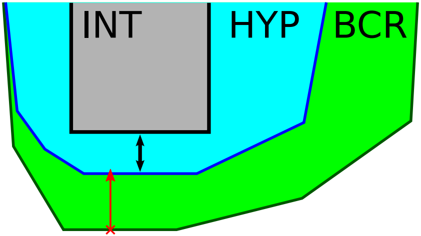
2. aim: compare gaps of HYP and BCR
$\rightarrow$ improve upper bound of BCR

Two birds, one stone...



1. solve BCR
2. compute solution to HYP from BCR
   $\rightarrow$ loss: β
3. use approximation for HYP:
   $\rightarrow$ loss: ln(4)

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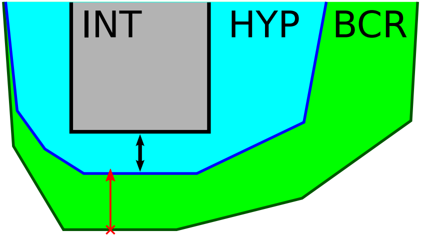

1. solve BCR
2. compute solution to HYP from BCR
   $\rightarrow$ loss: β
3. use approximation for HYP:
   $\rightarrow$ loss: $\ln(4)$

+ efficient algorithm
– total loss: $\beta \ln(4)$

but:

if $\beta < 2/\ln(4)$
then BCR gap < 2

Comparing the gaps: known results

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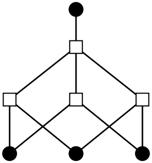

$$\frac{\text{HYP opt}}{\text{BCR opt}} = \frac{12}{11}$$

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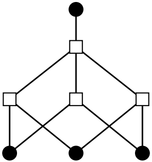

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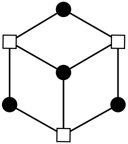


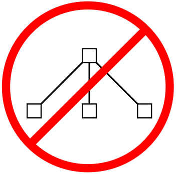
quasi-bipartite

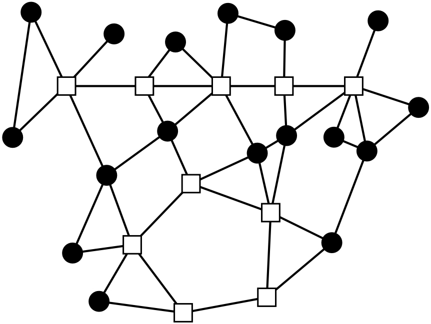
[Chakrabarty et al. 2010]
[Fung et al. 2012]
[Goemans et al. 2012]

Equal gaps: new results

Theorem

In every **Steiner claw-free** *instance, BCR gap $=$ HYP gap.*





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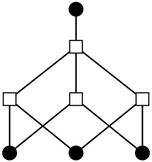

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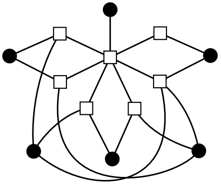
Theorem

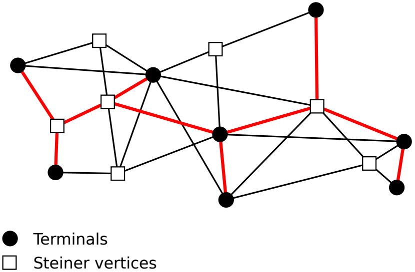
It is NP-hard to decide whether BCR opt $=$ HYP opt (even on instances with only one Steiner star).



BCR: undirected version

equivalent LP

[Goemans, Myung 1993]



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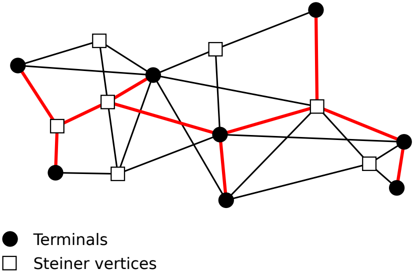


notation:

- $E(S)$: induced edges of S
- $y_{\max}(S) = \max_{v \in S} y_v$

$$\min \sum_{e \in E} z_e \operatorname{cost}(e) \quad \text{s.t.}$$

$$\sum_{e \in E(S)} z_e \leq \sum_{v \in S} y_v - y_{\max}(S) \qquad \forall S \subseteq V \qquad \text{(no cycles)}$$

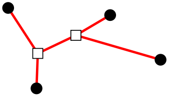
$$\sum_{e \in E} z_e = \sum_{v \in V} y_v - 1 \qquad \text{(connectedness)}$$

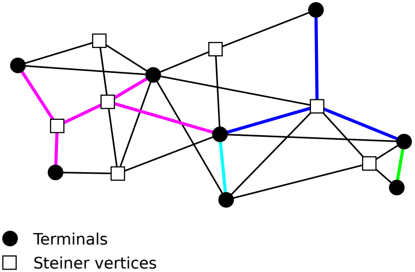
$$y_t = 1 \qquad \forall t \in R \qquad \text{(terminals in tree)}$$

$$y_v, z_e \geq 0 \qquad \forall v \in V, e \in E$$

Hypergraphic relaxation

based on *full components*:





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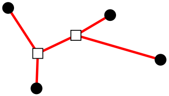
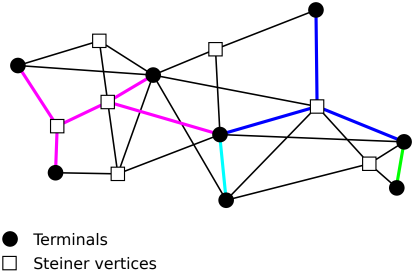




notation:

- $R(C)$: terminals in C
- $(a)^+ = \max\{0, a\}$

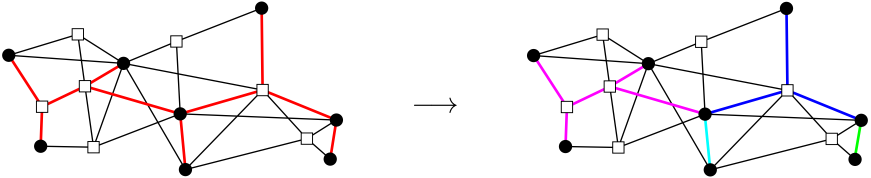
$$\min \sum_{C \in \mathcal{K}} x_C \operatorname{cost}(C) \quad \text{s.t.}$$

$$\sum_{C \in \mathcal{K}} x_C (|R(C) \cap S| - 1)^+ \leq |S| - 1 \qquad \forall S \subseteq R \qquad \text{(no cycles)}$$

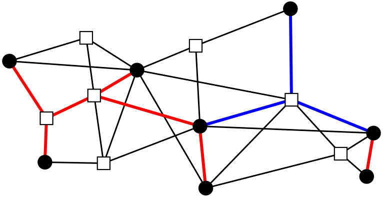
$$\sum_{C \in \mathcal{K}} x_C (|R(C)| - 1)^+ = |R| - 1 \qquad \text{(connectedness)}$$

$$x_C \geq 0 \qquad \forall C \in \mathcal{K}$$

From BCR to HYP



BCR HYP



1. identify component C of support
2. $y_v \to y_v - \varepsilon$, $\forall$ Steiners of C
3. $z_e \to z_e - \varepsilon$, $\forall$ edges of C
4. $x_C \to \varepsilon$
5. repeat

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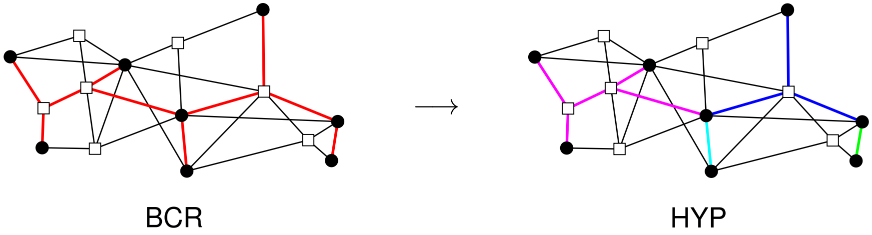

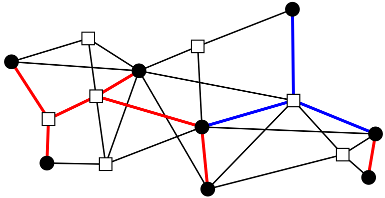

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Constant cost: $$\sum_{e \in E(C)} z_e \operatorname{cost}(e) = x_C \operatorname{cost}(C)$$

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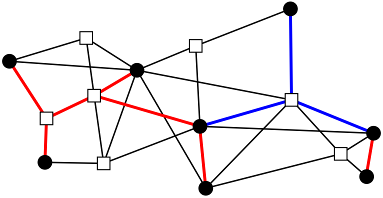

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Bottleneck: **tight set** S

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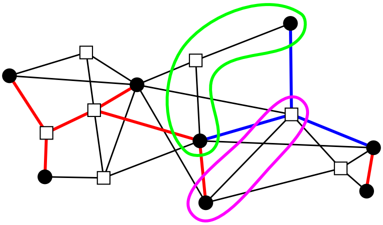

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Bottleneck: **tight set** S

Lemma

An iteration succeeds if for every tight set S intersecting C,

1. *C is connected in S, and*
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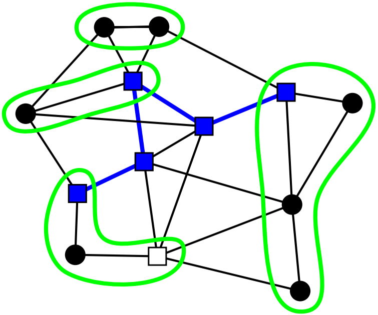

Identifying a component:

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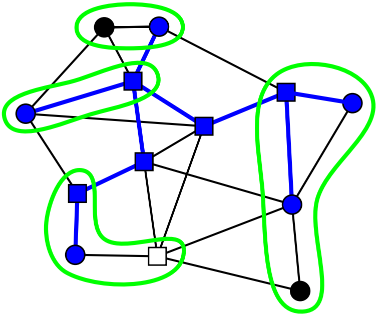

Identifying a component:

1. add neighboring Steiners, s.t. all tight sets are connected
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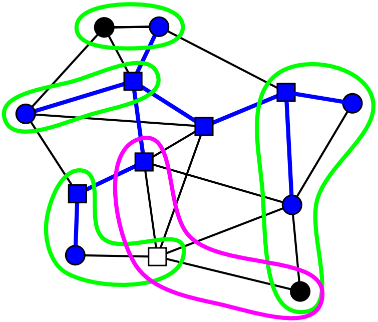


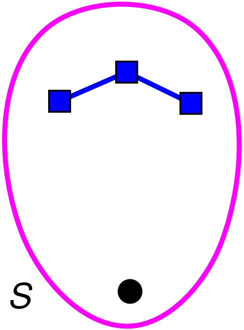
If the iteration fails, $\exists$ *demanding set*:

tight set intersecting C
s.t. maximizer not in C.

Demanding sets and blocked edges

A demanding set S has

- **maximizers** $\notin C$
- connected Steiners $\subseteq C$



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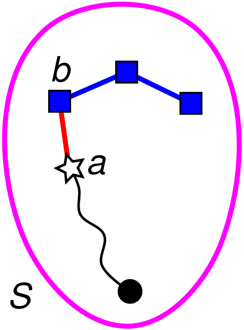

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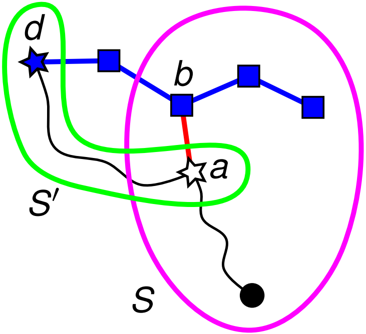

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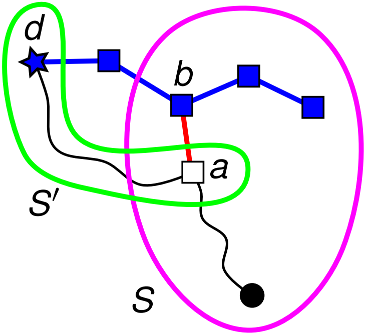

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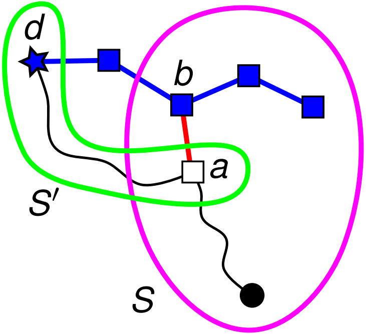

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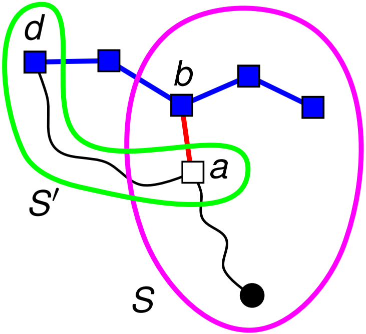

Demanding sets and blocked edges

Lemma

b is connected to a maximizer of S in $S \setminus S'$.

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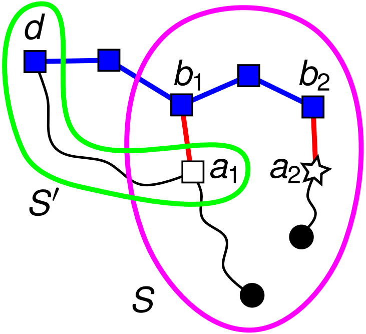

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  $a_1 \neq a_2$ are Steiners
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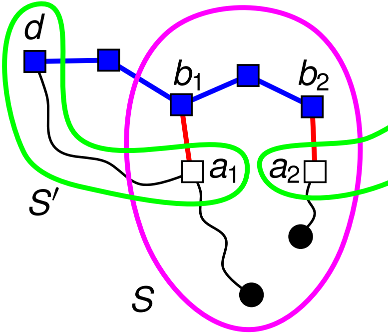

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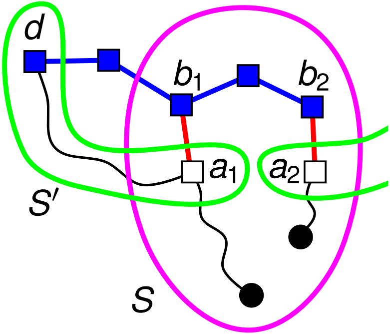


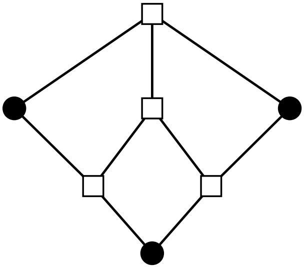
Theorem

In every **Steiner claw-free** *instance, BCR gap = HYP gap.*

Quo vadis?

Conjecture

If the following minor does not exist, then BCR gap $=$ HYP gap.



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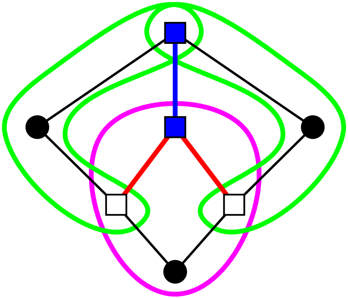
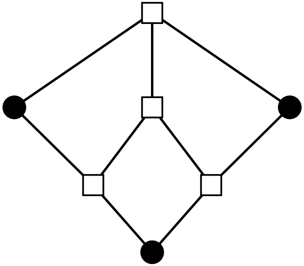


Quo vadis?

Conjecture

If the following minor does not exist, then BCR gap $=$ HYP gap.

$$\frac{\text{HYP opt}}{\text{BCR opt}} = \frac{16}{15}$$



Thanks!