Carving-width, tree-width and area-optimal planar graph drawing

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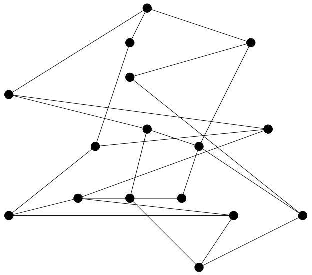
May 5, 2014

- Given: A graph $G = (V, E)$.



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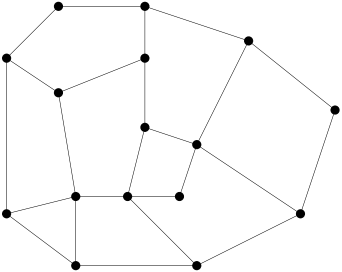

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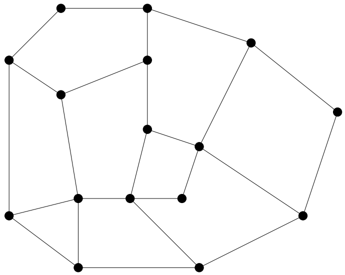

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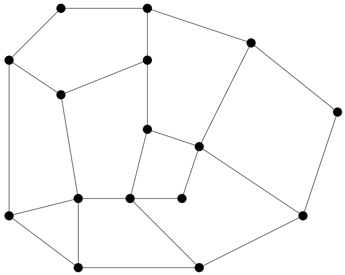
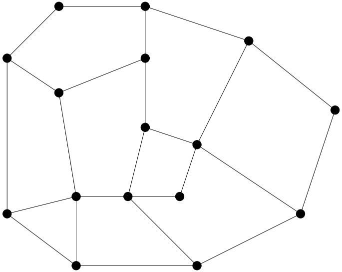


This talk: Straight-line drawing, planar graph, no crossing.

Previous work

- Every planar graph has a planar straight-line drawing. (Wagner'36, Fáry'48, Stein'50)



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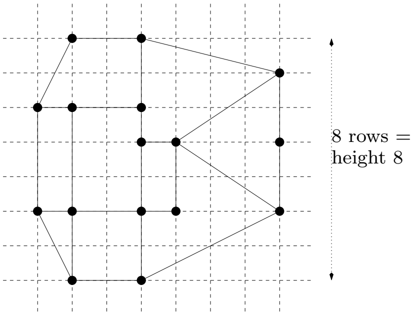

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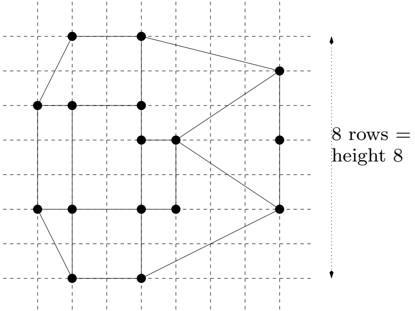

Theorem (de Fraysseix, Pach, Pollack 1990; Schnyder 1990)

Every planar graph can be drawn in an $O(n) \times O(n)$-grid.

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Many planar graph drawing results since:

- Drawing in an $n \times n$-grid (Schnyder 1990)
- Drawing in a $\frac{2}{3}n \times O(n)$-grid (Chrobak, Nakano 1994)
- Drawing in area $\frac{8}{9}n^2$ (Brandenburg 2008)
- 4-connected planar graphs in area $\frac{1}{4}n^2$ (Miura et al. 1999)
- Trees in $O(n \log n)$ area (easy).
- Outer-planar graphs in $O(n \log n)$ area (B. 2002)
- Series-parallel graphs in $O(n^{1.5})$ area (B. 2009)
- and many more like that....

Area-optimal planar graph drawing?

- Most graph drawing results have the form:

  Algorithm draws graph in class X with area $f(n)$.
  Some graph in class X needs area $\Omega(f(n))$.

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  Some graph in class X needs area $\Omega(f(n))$.

- Very few graph drawing results of the form:

  Algorithm draws G with optimal area for G.
  (Or at least within constant factor.)

Area-optimal planar graph drawing?

Problem

DRAWOPTAREA: *Given a planar graph G and a constant A, does G have a planar straight-line drawing of area at most A?*

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DrawOptArea: *Given a planar graph G and a constant A, does G have a planar straight-line drawing of area at most A?*

- NP-hard (Krug & Wagner 2007)

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  (planar 3-tree, maximal chordal planar, ...)

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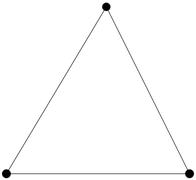

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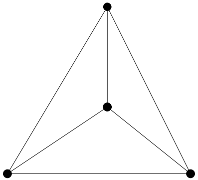

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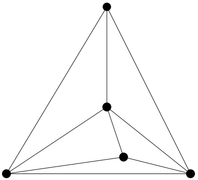

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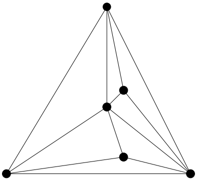

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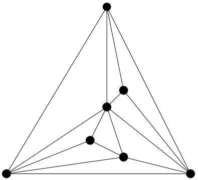

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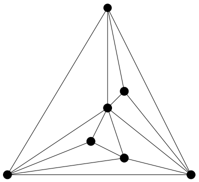

(planar 3-tree, maximal chordal planar, ...)

Why are Apollonian networks easy?

- treewidth 3?
- faces are triangles?
- both?

Treewidth

Definition

Treewidth $tw(G) =$

$$\min\{k\colon\ G \text{ has chordal super-graph with clique-size } k\} - 1$$

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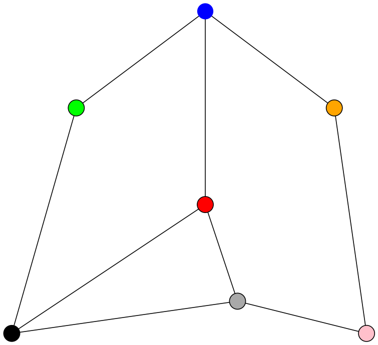

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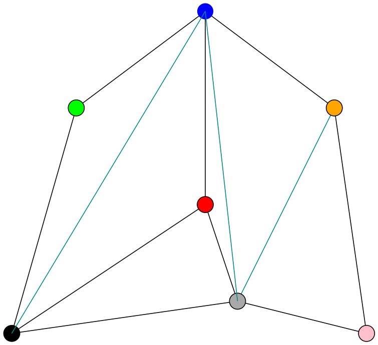

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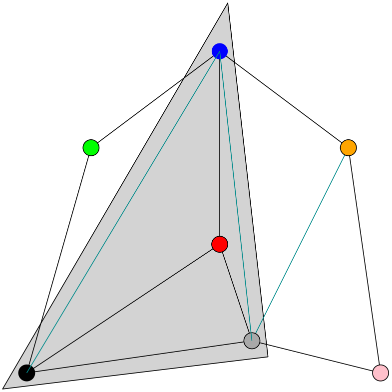

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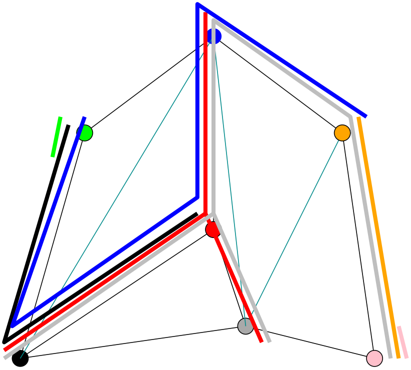

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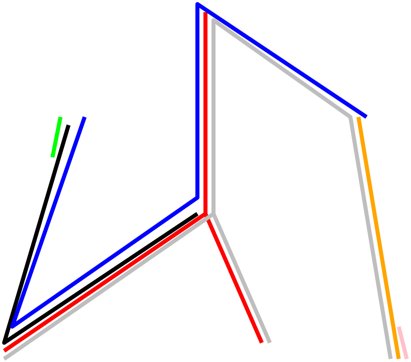

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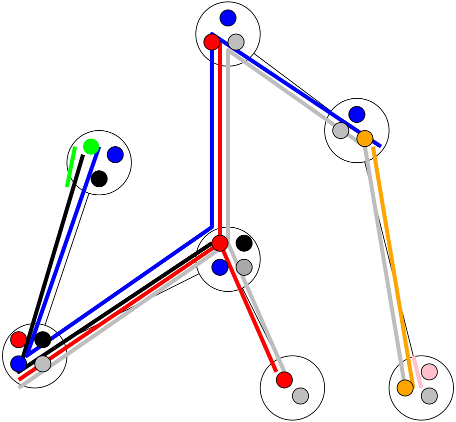

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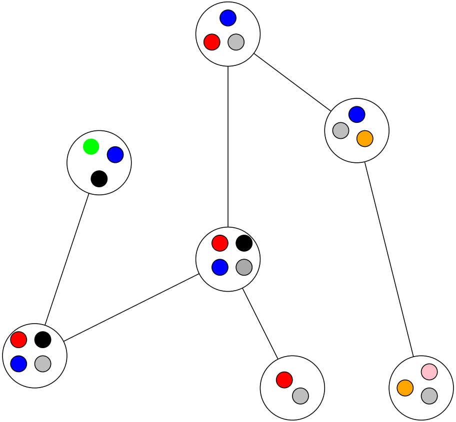

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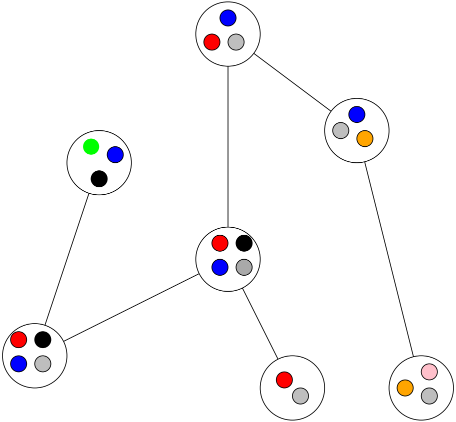

- G has bounded treewidth $\Rightarrow$ many NP-hard problems become polynomial (FPT).
- Often a first step towards developing a PTAS.

Planar graph drawing and treewidth

DrawOptArea is NP-hard. Polynomial if treewidth bounded?

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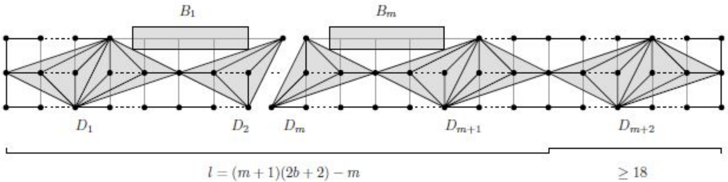

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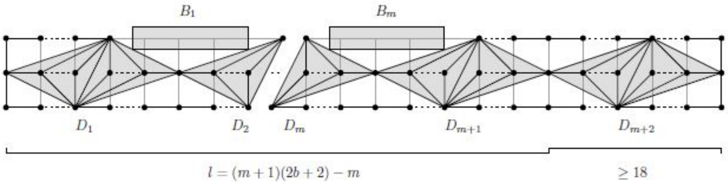

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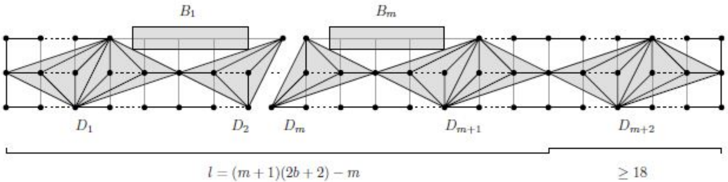

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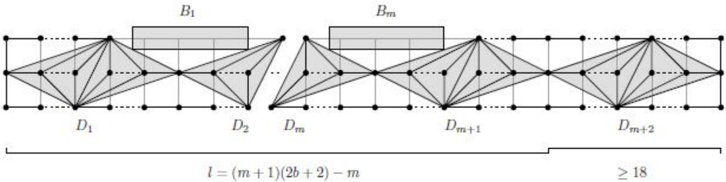

(Reduction-graph in [KW07] for NP-hardness of DrawOptArea.)

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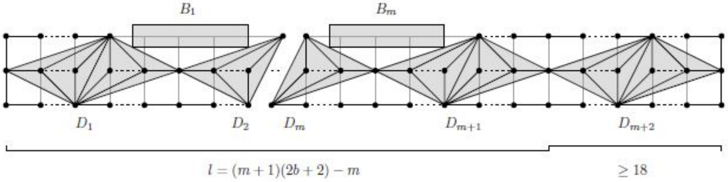
- Well-known: then treewidth ≤ 3.



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- But this graph was disconnected.
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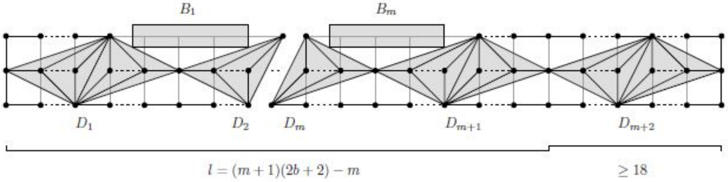
Planar graph drawing and treewidth



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DRAWOPTAREA *is NP-hard even for a 3-connected planar graph with treewidth at most 8.*

Planar graph drawing and treewidth

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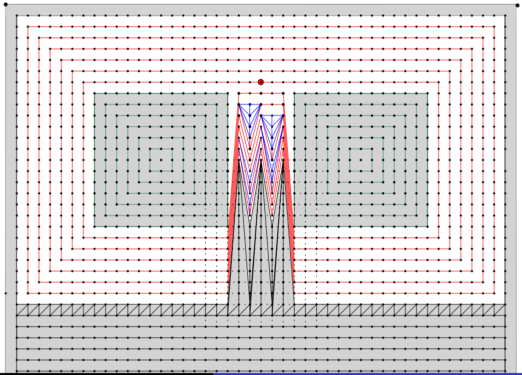

Back to: Why is DrawOptArea easy on Apollonian networks?

- constant treewidth?
- faces are triangles?
- both?

Planar graph drawing and triangulated graphs

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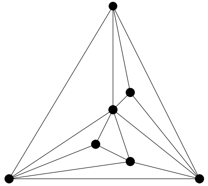
Definition (Planar triangulated graph)

Planar graph where all faces (including outer-face) are triangles.



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DRAWOPTAREA *is NP-hard even for planar graphs that are one edge away from being triangulated.*

Planar graph drawing and triangulated graphs

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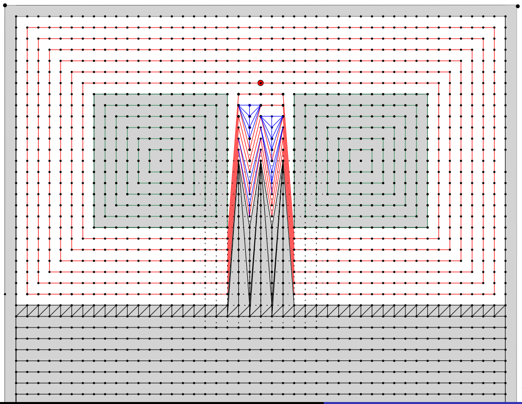

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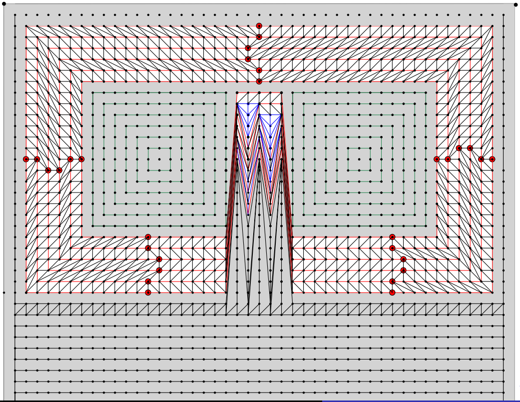

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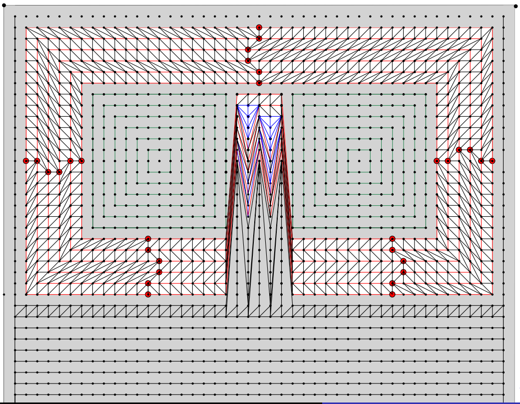
NP-hard for triangulated? Conjectured yes.



Planar graph drawing and carving width

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Theorem (based on B., Vatshelle 2012)

If G is a plane graph with bounded treewidth and bounded face-degrees, then DRAWOPTAREA *is polynomial.*

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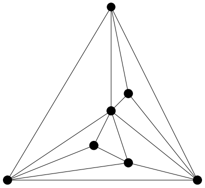

Look at dual graph:

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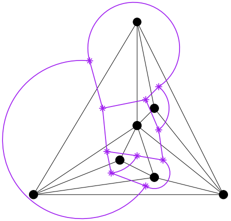


Look at dual graph:

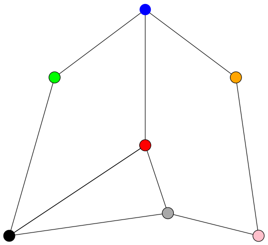
- bounded face degrees $\Leftrightarrow$ bounded maximum degree in dual
- bounded treewidth $\Leftrightarrow$ bounded treewidth in dual

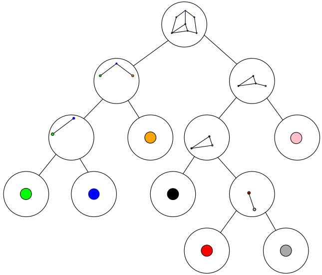
$\Rightarrow$ dual has bounded <u>carving width</u>

Carving-width

Definition (Carving decomposition)

Recursive vertex-partitioning (always split into two groups)





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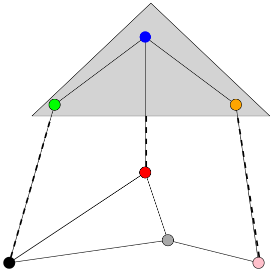
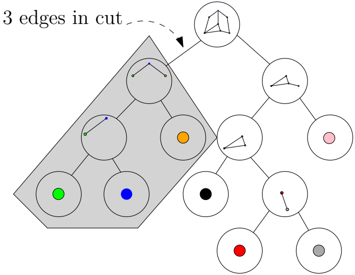


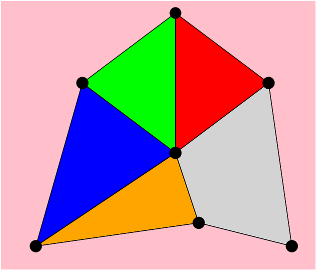


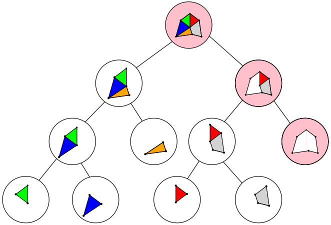
Definition

Width of carving decomposition: maximum # edges in cut at arc.
Carving width: Smallest possible width of carving decomposition.

Definition (Carving decomposition of dual)

Recursive face-partitioning (always split into two groups)





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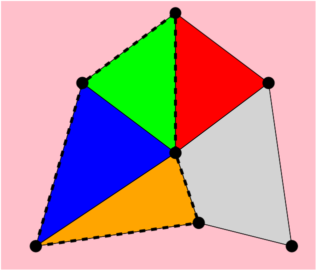
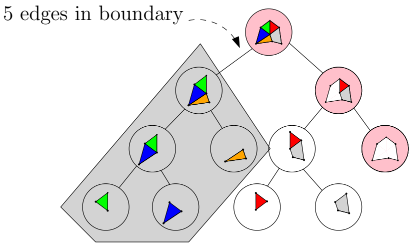




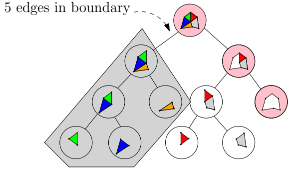
Carving width of dual: maximum # edges in boundary.

Planar graph drawing and carving width

Theorem (B. 2014, based on B., Vatshelle 2012)

If G is a plane graph whose dual has bounded carving width, then DrawOptArea *is polynomial.*

Idea: Dynamic programming.



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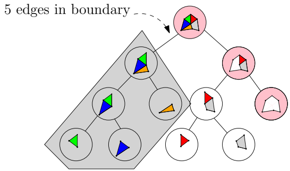
Idea: Dynamic programming.

- G_a: graph "below" arc a.
- B_a: boundary-vertices of G_a
- π_a: mapping from B_a to points in $W \times H$-grid.



$$M(a, \pi) = TRUE \text{ if we can draw } G_a \text{ planar with } B_a \text{ at } \pi(B_a)$$

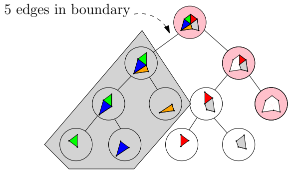
Planar graph drawing and carving width

Theorem (B. 2014, based on B., Vatshelle 2012)

If G is a plane graph whose dual has bounded carving width, then DrawOptArea *is polynomial.*

Idea: Dynamic programming.

- G_a: graph "below" arc a.
- B_a: boundary-vertices of G_a
- π_a: mapping from B_a to points in $W \times H$-grid.



$$M(a, \pi) = TRUE \text{ if we can draw } G_a \text{ planar with } B_a \text{ at } \pi(B_a)$$

- Compute M (for all $W \cdot H \leq A$) bottom-up in decomposition.
- G has drawing $\Leftrightarrow$ TRUE at root for some π, W, H.
- Run-time $O^*(A^{\frac{3}{2}cw(G^*)})$.

Area-optimal planar graph drawing

Planar embedding	Fixed	Free		
constant treewidth	NPC [KW07]			
constant face-degrees				
constant vertex-degrees				
constant treewidth and face-degrees				
constant treewidth and vertex-degrees				

Area-optimal planar graph drawing

Planar embedding	Fixed	Free		
constant treewidth	NPC [KW07]	NPC		
constant face-degrees				
constant vertex-degrees	NPC	NPC		
constant treewidth and face-degrees				
constant treewidth and vertex-degrees	NPC	NPC		

Area-optimal planar graph drawing

Planar embedding	Fixed	Free		
constant treewidth	NPC [KW07]	NPC		
constant face-degrees	NPC	NPC		
constant vertex-degrees	NPC	NPC		
constant treewidth and face-degrees				
constant treewidth and vertex-degrees	NPC	NPC		

Area-optimal planar graph drawing

Planar embedding	Fixed	Free		
constant treewidth	NPC [KW07]	NPC		
constant face-degrees	NPC	NPC		
constant vertex-degrees	NPC	NPC		
constant treewidth and face-degrees	P			
constant treewidth and vertex-degrees	NPC	NPC		

Area-optimal planar graph drawing

Planar embedding	Fixed	Free		
constant treewidth	NPC [KW07]	NPC		
constant face-degrees	NPC	NPC		
constant vertex-degrees	NPC	NPC		
constant treewidth and face-degrees	P	NPC		
constant treewidth and vertex-degrees	NPC	NPC		

Area-optimal planar graph drawing

Convex drawing	No		Yes	
Planar embedding	Fixed	Free	Fixed	Free
constant treewidth	NPC [KW07]	NPC		
constant face-degrees	NPC	NPC		
constant vertex-degrees	NPC	NPC		
constant treewidth and face-degrees	P	NPC		
constant treewidth and vertex-degrees	NPC	NPC		

DrawConvexOptArea: Only consider drawings where all faces (including outer-face) are convex.

Area-optimal planar graph drawing

Convex drawing	No		Yes	
Planar embedding	Fixed	Free	Fixed	Free
constant treewidth	NPC [KW07]	NPC		
constant face-degrees	NPC	NPC	NPC	NPC
constant vertex-degrees	NPC	NPC	NPC	NPC
constant treewidth and face-degrees	P	NPC		
constant treewidth and vertex-degrees	NPC	NPC		

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Area-optimal planar graph drawing

Convex drawing	No		Yes	
Planar embedding	Fixed	Free	Fixed	Free
constant treewidth	NPC [KW07]	NPC		
constant face-degrees	NPC	NPC	NPC	NPC
constant vertex-degrees	NPC	NPC	NPC	NPC
constant treewidth and face-degrees	P	NPC		
constant treewidth and vertex-degrees	NPC	NPC	P	

DRAWCONVEXOPTAREA: Only consider drawings where all faces (including outer-face) are convex.

Theorem (based on B.,Vatshelle 2012)

If G is a plane graph that has bounded carving width, then DRAWCONVEXOPTAREA *is polynomial.*

Area-optimal planar graph drawing

Convex drawing	No		Yes	
Planar embedding	Fixed	Free	Fixed	Free
constant treewidth	NPC [KW07]	NPC	P	P
constant face-degrees	NPC	NPC	NPC	NPC
constant vertex-degrees	NPC	NPC	NPC	NPC
constant treewidth and face-degrees	P	NPC	P	P
constant treewidth and vertex-degrees	NPC	NPC	P	P

DrawConvexOptArea: Only consider drawings where all faces (including outer-face) are convex.

Theorem (~~based on B.,Vatshelle 2012~~ B. 2014)

If G is a ~~plane~~planar graph that has bounded ~~carving width~~ treewidth, then DrawConvexOptArea *is polynomial.*

Some open problems

- Is DrawOptArea polynomial for other graph classes?

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References:

- T. Biedl, M. Vatshelle, The point-set embeddability problem for plane graphs, SoCG 2012, to appear in IJCGA.
- T. Biedl, On area-optimal planar drawings, to appear at ICALP 2014.