COMPLEMENTED SUBSPACES OF THE GROUP VON NEUMANN ALGEBRAS

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Outline of Talk

1. Locally compact group

2. Fourier algebra of a group

3. Invariant complementation property of the group von Neumann algebra

4. Fixed point sets of power bounded elements in $VN(G)$

5. Natural projections

1. Locally compact groups

A topological group $(G, \mathcal{T})$ is a group G with a Hausdorff topology $\mathcal{T}$ such that

(i) $G \times G \to G$

$(x, y) \to x \cdot y$

(ii) $G \to G$

$x \to x^{-1}$

are continuous. G is locally compact if the topology $\mathcal{T}$ is locally compact i.e. there is a basis for the neighbourhood of the identity consisting of compact sets.

Ex: G_d, $\mathbb{R}^n$, $(E, +)$, $\mathbb{T}$, $\mathbb{Q}$, $GL(2, \mathbb{R})$, $E =$ Banach space, $\mathbb{T} = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$

$CB(G) =$ bounded complex-valued continuous functions $f : G \to \mathbb{C}$

$\|f\|_u =$ sup $\{|f(x)| : x \in G\}$ $f \in CB(G)$, let $(\ell_a f)(x) = f(ax)$, $a, x \in G$.

$LUC(G) =$ bounded left uniformly continuous functions on G

$= \{f \in CB(G); a \to \ell_a f$ from G to $\big(CB(G), \|\cdot\|\big)$ is continuous$\}$

G is **amenable** if $\exists\, m \in LUC(G)^*$ such that

$m \geq 0, \quad \|m\| = 1$ and

$m(\ell_a f) = m(f)$ for all $a \in G, \quad f \in LUC(G)$.

Theorem (M.M. Day - T. Mitchell). *Let G be a topological group. Then G is amenable $\Longleftrightarrow G$ has the following fixed point property:*

> *Whenever $\mathcal{G} = \{T_g;\, g \in G\}$ is a continuous representation of G as continuous affice maps on a compact convex subset K of a separated locally convex space, then there exist $x_0 \in K$ such that $T_g(x_0) = x_0$ for all $g \in G$.*

Amenable Groups:

- abelian groups
- solvable groups
- compact groups
- $U\big(B(\ell_2)\big)=$ group of unitary operators on ℓ_2 with the strong operator topology where

$$\ell_2 = \{(\alpha_n) : \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty\}$$

$\mathbb{F}_2$ – not amenable

Let G be a locally compact group and λ be a fixed left Haar measure on G.

$L^1(G) =$ group algebra of G i.e. $f : G \mapsto \mathbb{C}$ measurable such that

$$\int |f(x)| d\lambda(x) < \infty$$

$$(f * g)(x) = \int f(y) g(y^{-1}x) d\lambda(y)$$

$$\|f\|_1 = \int |f(x)| d\lambda(x)$$

$\big(L^1(G), *\big)$ is a Banach algebra i.e. $\|f * g\| \le \|f\| \, \|g\|$ for all $f, g \in L^1(G)$

$L^\infty(G) =$ essentially bounded measurable functions on G.

$\|f\|_\infty =$ ess - sup norm. $= \inf \big\{ M : \{x \in G; |f(x)| > M \text{ is a locally null set}\} \big\}$

$L^\infty(G)$ is a commutative C^*-algebra containing $CB(G)$

$$L^1(G)^* = L^\infty(G) : \langle f, h \rangle = \int f(x) h(x) d\lambda(x)$$

$G =$ locally compact **abelian** group then $L^1(G)$ is a **commutative** Banach algebra.

A complex function γ on G is called a **character** if γ is a homomorphism of G into $(\mathbb{T}, \cdot)$.

$$\widehat{G} = \text{ all continuous characters on } G$$
$$\subseteq L^\infty(G) = L^1(G)^*.$$

If $\gamma \in \Gamma, \quad f \in L^1(G)$,

$$\langle \gamma, f \rangle = \widehat{f}(\gamma) = \int_G f(x)(-x, \gamma) dx.$$

Then $\langle \gamma, f * g \rangle = \langle \gamma, f \rangle \ \langle \gamma, g \rangle$ for all $f, g \in L^1(G)$. Hence γ defines a non-zero multiplicative linear functional on $L^1(G)$. Conversely every non-zero multiplicative linear functional on $L^1(G)$ is of this form:

$$\sigma\big(L^1(G)\big) \cong \widehat{G}.$$

Example

$$G = \mathbb{R} \qquad \widehat{G} = \mathbb{R}$$

$$G = \mathbb{T} \qquad \widehat{G} = \mathbb{Z}$$

$$G = \mathbb{Z} \qquad \widehat{G} = \mathbb{T}.$$

Equip $\widehat{G}$ with the weak*-topology from $L^1(G)^*$ (or the topology of uniform convergence on compact sets). Then

$$\widehat{G} \text{ with } \textbf{product}: \quad (\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x)$$

is a locally compact **abelian** group.

- **Pontryagin Duality Theorem:** $\widehat{\widehat{G}} \cong G$

  For $f \in L^1(G)$, $\quad \widehat{f} : \widehat{G} \to \mathbb{C}$

$$\widehat{f}(\gamma) = \int_G f(x)(-x, \gamma)dx = \langle f, \gamma \rangle$$

- $A(\widehat{G}) = \{\widehat{f};\ f \in L^1(G)\} \subseteq C_0(\widehat{G}) =$ functions in $CB(\widehat{G})$ vanishing at infinity.

- $\theta : f \to \widehat{f}$ is an algebra homomorphism from $L^1(G)$ into a subalgebra of $C_0(\widehat{G})$.

- $\big(A(\widehat{G}), \|\cdot\|\big)$ $\quad \|\widehat{f}\| = \|f\|_1$ is a commutative Banach algebra with spectrum $\widehat{G}$.

  $A(\widehat{G}) =$ Fourier algebra of $\widehat{G}$.

2. Fourier algebra of a group

$G =$ locally compact group

A continuous unitary representation of G is a pair: $\{\pi, H\}$, where $H =$ Hilbert space and π is a continuous homomorphism from G into the group of unitary operators on H such that for each ξ, $n \in H$,

$$x \longrightarrow \langle \pi(x)\xi, n \rangle$$

is continuous.

$$L^2(G) = \quad \text{all measurable} \quad f : G \to \mathbb{C}$$

$$\int |f(x)|^2 d\lambda(x) < \infty$$

$$\langle f, g \rangle = \int f(x)\, \overline{g(x)}\, d\lambda(x)$$

$$L^2(G) \quad \text{is a Hilbert space.}$$

Left regular representation:

$$\{\rho, L^2(G)\},$$

$$\rho : G \mapsto B\big(L^2(G)\big),$$

$$\rho(x)h(y) = h(x^{-1}y),\ x \in G,\ h \in L^2(G).$$

$$
\begin{aligned}
G = &\quad \text{locally compact group} \\
A(G) = &\quad \text{subalgebra of} \quad C_0(G) \\
&\quad \text{consisting of all functions} \quad \phi : \\
\phi(x) = &\langle \rho(x)h, k \rangle, \quad h, k \in L^2(G) \\
\rho(x)h(y) = &h(x^{-1}y) \\
\|\phi\| = &\sup \left\{ \Big| \sum_{i=1}^{n} \lambda_i \phi(x_i) \Big| : \Big\| \sum_{i=1}^{n} \lambda_i \rho(x_i) \Big\| \le 1 \right\} \\
\ge &\|\phi\|_\infty .
\end{aligned}
$$

P. Eymard (1964):

$$\begin{aligned} A(G)^* &= VN(G) \\ &= \text{ von Neumann algebra in } \quad \mathcal{B}\big(L^2(G)\big) \\ &\qquad \text{generated by } \quad \{\rho(x) : x \in G\} \\ &= \overline{\langle \rho(x) : x \in G \rangle}^{\text{WOT}} = \{\rho(x); x \in G\} \quad \text{(second commutant)} \end{aligned}$$

If G is abelian, then

$$A(G) \cong L^1(\widehat{G}), \qquad VN(G) \cong L^\infty(\widehat{G}).$$

- $A(G)$ is called the *Fourier algebra of* G.
- $VN(G)$ is called the *group von Neumann algebra* of G.
- $VN(G)$ can be viewed as *non-commutative function space* on $\widehat{G}$ when G is non-abelian.

Theorem (P. Eymard 1964)**.** *For any* G, $A(G)$ *is a commutative Banach algebra with spectrum* G.

Theorem (H. Leptin 1968)**.** *For any* G, $A(G)$ *has a bounded approximate identity if and only if* G *is amenable.*

Theorem (M. Walters 1970)**.** *Let* G_1, G_2 *be locally compact groups. If* $A(G_1)$ *and* $A(G_2)$ *are isometrically isomorphic, then* G_1 *and* G_2 *are either isomorphic or anti-isomorphic.*

3. Invariant complementation property of the group von Neumann algebra

Theorem (H. Rosenthal 1966)**.** *Let G be a locally compact* **abelian** *group, and X be a weak*-closed translation invariant subspace of $L^\infty(G)$. If X is complemented in $L^\infty(G)$, then X is invariantly complemented i.e. X admits a translation invariant closed complement (or equivalently X is the range of a continuous projection on $L^\infty(G)$ commuting with translations).*

Theorem (Lau, 1983)**.** *A locally compact group G is amenable if and only if every weak*-closed left translation invariant subalgebra M which is closed under conjugation in $L^\infty(G)$ is invariantly complemented.*

For $T \in VN(G)$, $\phi \in A(G)$, define

$$\phi \cdot T \in VN(G) \quad \text{by}$$

$$\langle \phi \cdot T, \psi \rangle = \langle T, \psi\phi \rangle, \quad \psi \in A(G).$$

$X \subseteq VN(G)$ is **invariant** if $\phi \cdot T \in X$ for all $\phi \in A(G)$, $T \in X$.

If G is abelian and $X \subseteq L^\infty(\widehat{G})$ is weak*-closed subspace of $L^\infty(\widehat{G})$, then X is translation invariant $\iff$

$$L^1(\widehat{G}) * X \subseteq X.$$

Hence: weak*-closed $A(G)$-invariant subspaces of $VN(G)$ $\leftrightarrow$ weak*-closed translation invariant subspaces of $L^\infty(\widehat{G})$.

Question: Let G be a locally compact group, and M be an invariant W^*-subalgebra (i.e. weak*-closed $*$-subalgebra) of $VN(G)$. Is M invariantly complemented?

Equivalently: Is there a continuous projection

$$P : VN(G) \underset{\text{onto}}{\longrightarrow} M \quad \text{such that} \quad P(\phi \cdot T) = \phi \cdot P(T)$$

for all $\phi \in A(G)$.

Yes: G-abelian (Lau, 83)

Losert-L(86): **Yes:** G compact, discrete.

Theorem (Losert-Lau. 1986)**.** *Let M be an invariant W^*-subalgebra of $VN(G)$ and*

$$\sum(M) = \{x \in G;\ \rho(x) \in M\}.$$

If $\sum(M)$ is a normal subgroup of G, then M is invariantly complemented.

Let H be a closed subgroup of G, and

$$VN_H(G) = \overline{\langle \rho(h) : h \in H \rangle}^{\text{WOT}} \subseteq VN(G).$$

Then $VN_H(G)$ is an invariant W^*-subalgebra of $VN(G)$.

Takesaki-Tatsuma (1971): If M is an invariant W^*-subalgebra of $VN(G)$, then

$$M = \overline{\langle \rho(x) : x \in H \rangle}^{W^*} = VN_H(G)$$

where $H = \Sigma(M)$. Hence there is a $1-1$ correspondence between closed subgroups H of G and invariant W^*-subalgebras of $VN(G)$.

$G \in$ [SIN] if there is a neighbourhood basis of the identity consisting of compact sets V, $x^{-1}Vx = V$ for all $x \in G$.

[SIN]-groups include: compact, discrete, abelian groups.

A locally compact group G is said to have the **complementation property** if every weak*-closed invariant W^*-subalgebra of $VN(G)$ is **invariantly complemented.**

Theorem (Kaniuth-Lau 2000)**.** *Every [SIN]-group has the complementation property.*

Converse is false: The Heisenberg group has the complementation property but it is not a SIN group.

For a closed subgroup $H < G$, let

$$P_1(G) = \{\phi \in P(G);\ \phi(e) = 1\}$$

$$P_H(G) = \{\phi \in P(G);\ \phi(h) = 1\ \forall\, h \in H\} \subseteq P_1(G)$$

$$P_H(G) \quad \text{is a commutative semigroup.}$$

We call H a *separating subgroup* if for any $x \in G \backslash H$, there exists $\phi \in P_H(G)$ such that $\phi(x) \neq 1$.

G is said to have the *separation property* if each closed subgroup of G is separating.

(**Lau-Losert**, 1986) The following subgroups H are always separating:

- H is open
- H is compact
- H is normal

(Forrest, 1992): Every SIN-group has the separation property.

Example 1: $G =$ affine group of the real line $= 2 \times 2$

matrices of form

$$\left\{ \begin{pmatrix} a & s \\ 0 & 1 \end{pmatrix} : \ a > 0, \ s \in \mathbb{R} \right\} \longleftrightarrow \{(a,s); \ a > 0, \ s \in \mathbb{R}\}$$

$$(a,s)(b,t) = (ab, \ s + at).$$

Let $H = \{(a,0); a > 0\}$. Then H is not separating.

Note: If $\phi \in P_H(G), \quad x, y \in G$

$$|\phi(xy) - \phi(x)\phi(y)|^2 \leq \big(1 - |\phi(x)|^2\big)(1 - |\phi(y)|^2\big).$$

Hence $\phi(h_1 x h_2) = \phi(x)$ (+)

$$\forall \, x \in G, \quad h_1, \, h_2 \in H.$$

For $t > 0, \quad x_t = (1, t)$

$$H(1, t)H = G^+ = \{(a, s);\ a > 0,\ s > 0\}.$$

Hence:

$$\phi(h_1 x_t h_2) = \phi(x_t) \quad \forall\ h_1, h_2 \in H$$

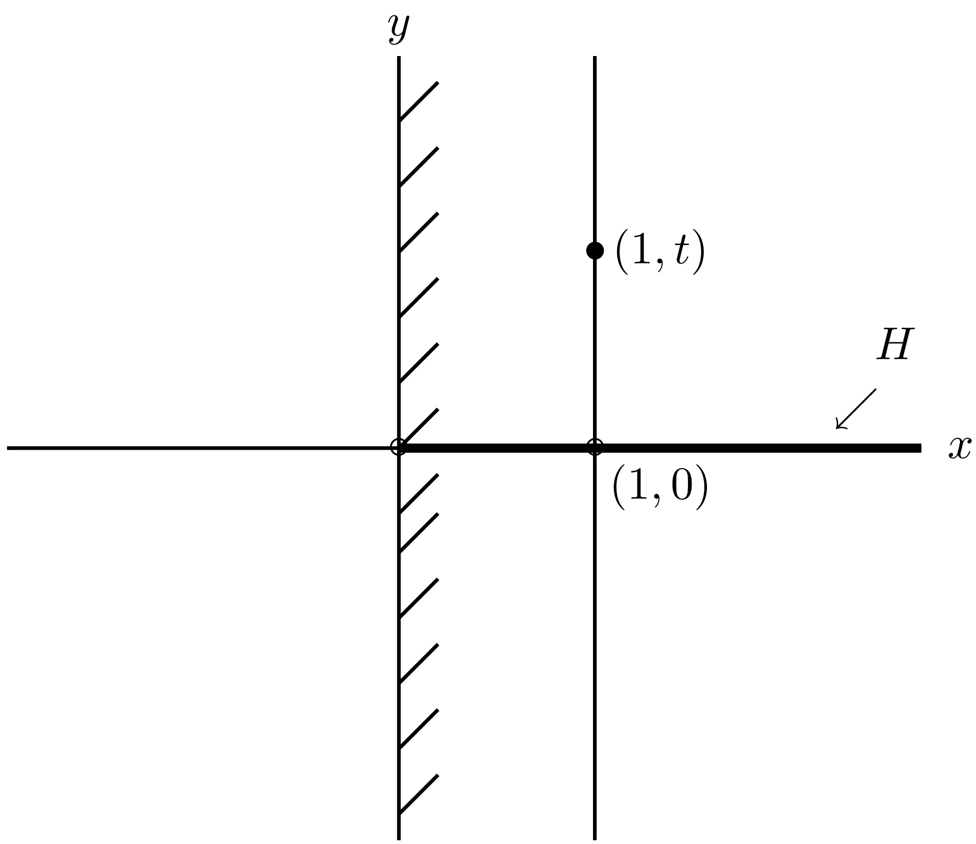
so by continuity, $t \to 0^+$

$$\phi(g) = 1 \quad \text{for all} \quad g \in G^+.$$

Similarly, by considering $t < 0$,

$$\phi(g) = 1 \quad \text{for all} \quad g \in G^-.$$

Consequently $\phi = 1$.



y
$(1,t)$
H
x
$(1,0)$

Example 2: $G =$ Heisenberg group

$$G = \text{ all } 3 \times 3 \text{ matrices}$$

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \longleftrightarrow (x, y, z)$$

$$\begin{aligned} &(x_1, y_1, z_1)(x_2, y_2, z_2) \\ &\quad = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2 + x_1 y_2) \end{aligned}$$

$$\begin{aligned} \text{Centre of } G &= Z(G) \\ &= \{(0, 0, t);\ t \in \mathbb{R}\} \end{aligned}$$

Let $H = \{(x, 0, 0);\ x \in \mathbb{R}\} < G$. Then H is not separating.

Let $\phi \in P_H(G)$. For $y \neq 0$, let $g_y = (0, y, 0)$. Then

$$\begin{aligned} \{hg_y h^{-1} g_y^{-1}; h \in H\} &= \{(0,0,t) : t \in \mathbb{R}\} \\ &= Z(G). \end{aligned}$$

Since
$$\begin{aligned} \phi(g_y) &= \phi(hg_y h^{-1}) \\ &= \phi\big(\underbrace{(hg_y h^{-1} g_y^{-1})}_{\in Z(G)} \cdot g_y\big) \end{aligned}$$

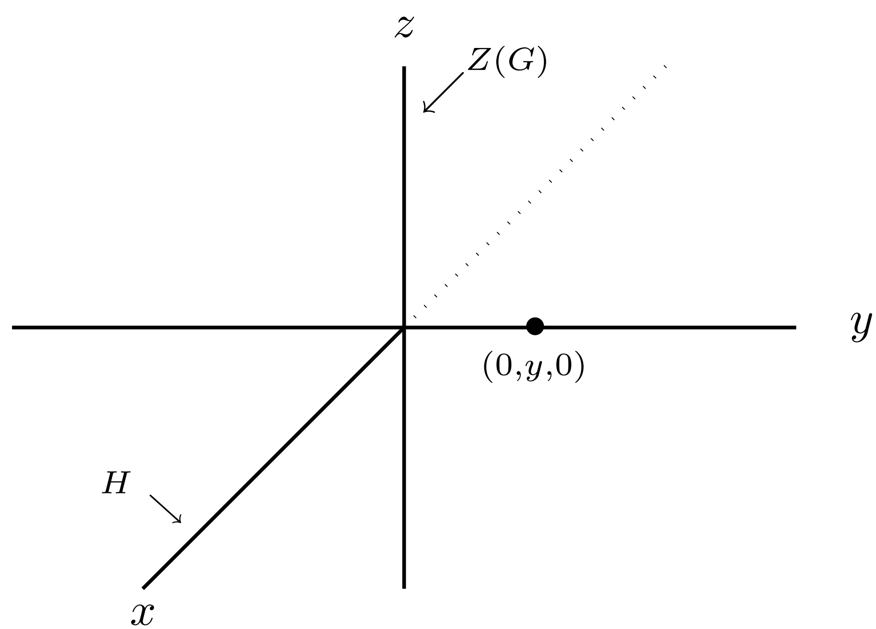
we obtain that

$$\phi(g_y) = \phi(g_y \cdot g) \quad \forall\, g \in Z(G)$$

$y \neq 0, \quad y \in \mathbb{R}$.

With $y \to 0$, we conclude that

$$\phi(g) = 1 \quad \forall\, g \in Z(G).$$


z
Z(G)
y
(0,y,0)
H
x

Theorem 7 (Kaniuth-Lau 2000)**.** (a) *For any locally compact group* G, *separation property implies invariant complementation property.*

(b) *Let* G *be a connected locally compact group. Then* G *has the separation property* $\iff G \in$ *[SIN].*

Losert (2008):

There is an example of a locally compact group G such that G has a compact open normal subgroup and every proper closed subgroup of G is compact (in particular, G is an IN-group) with the separation property and hence the invariant complementation property but G is not a SIN-group.

Theorem 1 (Forrest, Kaniuth, Spronk and Lau, 2003)**.** *Let* G *be an amenable locally compact group. Then* G *has the invariant complementation property.*

Open Problem 1: Does every locally compact group have the invariant complementation property?

4. Fixed point sets of power bounded elements in $VN(G)$

G–locally compact group

$P(G) =$ continuous positive definite functions on G

i.e. all continuous $\phi : G \to \mathbb{C}$ such that

$$\sum \lambda_i \overline{\lambda}_j \phi(x_i x_j^{-1}) \geq 0, \qquad \begin{matrix} x_1, \ldots, x_n \in G, \\ \lambda_i, \ldots, \lambda_n \in \mathbb{C} \end{matrix}$$

i.e. the $n \times n$ matrix $\big(\phi(x_i x_j^{-1})\big)$ is positive

$\phi \in P(G) \Longleftrightarrow$ there exists a continuous unitary representation $\{\pi, \mathcal{H}\}$ of G, $\eta \in \mathcal{H}$, such that

$\phi(x) = \langle \pi(x)\eta, \eta \rangle, \quad x \in G.$

Let $B(G) = \langle P(G) \rangle \subseteq CB(G)$ (Fourier Stieltjes algebra of G)

Equip $B(G)$ with norm $\|u\| = \sup \left\{ \left| \int f(t)u(t)dt \right|; \, f \in L^1(G) \text{ and } |||f||| \leq 1 \right\}$

where

$$|||f||| = \sup\{\|\pi(f)\|; \, \{\pi, H\} \quad \text{continuous unitary representation of } G\}$$

- When G is *amenable*, then $|||f||| = \|\rho(f)\|$, where ρ is the left regular representation of G.
- When G is abelian, $B(G) \cong M(\widehat{G})$ (measure algebra of $\widehat{G}$).

For a discrete group D, let $R(D)$ denote the Boolean ring of subsets of D generated by all left cosets of subgroups of D.

$$\text{Let} \quad R_c(G) = \{E \in R(G_d) : \; E \text{ is closed in } G\}$$

$G_d =$ denote G with the discrete topology.

Theorem (J. Gilbert, B. Schreiber, B. Forrest). $E \in R_c(G) \Longleftrightarrow$ $E = \bigcup_{i=1}^{n} \big(a_i H_i \backslash \bigcup_{j=1}^{m_i} b_{i,j} K_{ij}\big)$, *where* $a_i, b_{i,j} \in G$, H_i *is a closed subgroup of* G *and* K_{ij} *is an open subgroup of* H_i .

Let G and H be groups. A map $\alpha : C \subseteq G \to H$ is called *affine* if C is a coset and for any $r, s, t \in C$,

$$\alpha(rs^{-1}t) = \alpha(r)\alpha(s)^{-1}\alpha(t).$$

A map $\alpha : Y \subseteq G \to H$ is called *piecewise affine* if

(i) there exist pairwise disjoint sets $Y_i \in \mathcal{R}(G)$, $i = 1, \ldots, n$, such that $Y = \bigcup_{i=1}^{n} Y_i$,

(ii) each Y_i is contained in a coset C_i on which there is an affine map $\alpha_i : C_i \to H$ such that $\alpha_i|_{Y_i} = \alpha|_{Y_i}$.

Theorem (Illie and Spronk 2005)**.** *Let* G *and* H *be locally compact groups with* G *amenable, and let* $\Phi : A(G) \to B(H)$ *be a completely bounded homomorphism. Then there is a continuous piecewise affine map* $\alpha : Y \subset H \to G$ *such that for each* h *in* H

$$\Phi_u(h) = \begin{cases} u\big(\alpha(h)\big) & \textit{if} \quad h \in Y, \\ 0 & \textit{otherwise.} \end{cases}$$

Lemma A. *Let* G *be a locally compact group and* u *a power bounded element of* $B(G)$ *such that* E_u *is open in* G. *Then* $u|_{E_u}$ *is a piecewise affine map from* E_u *into* $\mathbb{T}$.

Proof. For $f \in B(\mathbb{T})$, define a function $\phi(f)$ on G by $\phi(f)(x) = f\big(u(x)\big)$ for $x \in E_u$ and $\phi(f)(x) = 0$ otherwise. Then $\phi(f)(u)$ is continuous since E_u is open and closed in G. Because $B(\mathbb{T}) = \widehat{\ell^1(\mathbb{Z})}$, we have

$$\sum_{n\in\mathbb{Z}} \check{f}(n)\overline{u}^{\,n} \in B(G),$$

where $\check{f}$ denotes the inverse Fourier transform of f, and

$$\phi(f)(x) = \sum_{n\in\mathbb{Z}} \check{f}(n)\,\overline{n(x)}^{\,n}$$

for all $x \in E_u$. Since $E_u \in \mathcal{R}_c(G)$, $1_{E_u} \in B(G)$, and therefore

$$\phi(f) = 1_{E_u} \cdot \sum_{n\in\mathbb{Z}} \check{f}(n)\,\overline{u}^{\,n} \in B(G).$$

Since fg is the inverse Fourier transform of $\check{f} * \check{g}$, it is straightforward to check that ϕ is a homomorphism from $B(\mathbb{T})$ into $B(G)$. Since ϕ is bounded and $B(\mathbb{T}) = \ell^1(\mathbb{Z})$ carries the MAX operator space structure, ϕ is actually completely bounded. It now follows from that there exists an affine map $\alpha : Y \subseteq G \to \mathbb{T}$ such that, for each $f \in B(\mathbb{T})$ and $x \in G$, $\phi(f)(x) = f\big(\alpha(x)\big)$ whenever $x \in Y$ and $\phi(f)(x) = 0$ otherwise. Here

$$Y = \{x \in G : \phi(f)(x) \neq 0 \quad \text{for some} \quad f \in B(\mathbb{T})\}.$$

It is then obvious that $Y = E_u$ and $\alpha = u|_{E_u}$ is piecewise affine. $\square$

For $\sigma \in B(G)$, $T \in VN(G)$, define $\sigma \cdot T \in VN(G)$

$$\langle \sigma \cdot T, \psi \rangle = \langle T, \sigma\psi \rangle, \quad \psi \in A(G).$$

$$\text{Let} \quad I_\sigma = \overline{\{\sigma\phi - \phi : \phi \in A(G)\}}^{\|\cdot\|}$$
$$\subseteq A(G).$$

Then

(i) I_σ is a closed ideal in $A(G)$

(ii) $I_\sigma^\perp = \{T \in VN(G) : \sigma \cdot T = T\}$ (σ-harmonic functionals on $A(G)$: Chu-Lau (2002)) is a weak*-closed invariant subspace of $VN(G)$.

If $u \in B(G)$, let

$$E_u = \{x \in G;\ |u(x)| = 1\} \quad \text{and}$$

$$F_u = \{x \in G;\ u(x) = 1\}.$$

Theorem (Kaniuth-Lau-Ülger 2010, JLMS)**.** *Let G be any locally compact group and $u \in B(G)$ be power bounded (i.e. $\sup\{\|x^n\|;\ n = 1, 2, \dots\} < \infty$). Then*

(a) *The sets E_u and F_u are in $R_c(G)$.*

(b) *The fixed point set of u in $VN(G) = \{T \in VN(G);\ u \cdot T = T\}$ is the range of a projection $P : VN(G) \to VN(G)$ such that $u \cdot P(T) = P(u \cdot T)$ for all $T \in VN(G)$. If G is amenable, then $\{T \in VN(G);\ u \cdot T = T\} = \overline{\langle \rho(x);\ x \in F_u \rangle}^{w^*}$.*

Note: When G is abelian, (a) is due to B. Schrieber.

Theorem (Kaniuth, Lau and Ülger, JFA 2011)**.** *Let* G *be a locally compact group and let* u *be a power bounded element of* $B(G)$*. Then there exist closed subsets* $F_1, \ldots, F_n$ *of* G *with the following properties:*

(1) $F_j \in \mathcal{R}_c(G)$, $1 \leq j \leq n$, and $E_u = \bigcup_{j=1}^{n} F_j$.

(2) *For each* $j = 1, \ldots, n$, *there exist a closed subgroup* H_j *of* G, $a_j \in G$, $\alpha_j \in \mathbb{T}$ *and a continuous character* γ_j *of* H_j *such that* $F_j \subseteq a_j H_j$ *and*

$$u(x) = \alpha_j \gamma_j (a_j^{-1} x)$$

for all $x \in F_j$.

Proof. Consider the group G equipped with the discrete topology. Let $i : G_d \to G$ denote the identity map. Then $u \circ i \in B(G_d)$ and $\|u \circ i\|_{B(G_d)} = \|u\|_{B(G)}$ and hence $u \circ i$ is power bounded. Therefore, by Lemma A there exist subsets S_i of G, subgroups L_i of G, $c_i \in G$ and affine maps $\beta_i : c_i L_i \to \mathbb{T}$, $i = 1, \dots, r$, with the following properties:

(1) $S_i \in \mathcal{R}(G_d)$ and $E_u = \bigcup_{i=1}^{n} S_i$;

(2) For each $i = 1, \dots, n$, $S_i \subseteq c_i L_i$ and $\beta_i|_{S_i} = u|_{S_i}$.

Now each S_i is of the form

$$\bigcup_{\ell=1}^{q} d_\ell \Big(M_\ell \backslash \bigcup_{k=1}^{q_\ell} e_{\ell k} N_{\ell k} \Big),$$

where $d_\ell, e_{\ell k} \in G$, the M_ℓ are subgroups of G and the $N_{\ell k}$ are subgroups of M_ℓ, $1 \le \ell \le q$, $1 \le k \le q_\ell$. Thus, by a further reduction step, we can assume that we only have to consider a set S of the form

$$S = a\Big(H \backslash \bigcup_{j=1}^{m} b_j K_j\Big) \subseteq bT,$$

where $b_j \subset H$ and the K_j are subgroups of H, and that there exists an affine map $\beta : bT \to \mathbb{T}$ such that $\beta|_S = u|_S$. Furthermore, we can assume that each K_j has infinite index in H because otherwise, for some j, H is a finite union of K_j-cosets, and therefore can be assumed to be simply a coset.

Now

$$H = (H \cap a^{-1}bT) \cup \bigcup_{j=1}^{n} b_j K_j \quad \text{and} \quad H \cap a^{-1}bT \neq \emptyset,$$

because otherwise at least one of the K_j has finite index in H. It follows that $H \cap a^{-1}bT = h(H \cap T)$ for some $h \in H$ and $H \cap T$ has finite index in H. So S is contained in a finite union of cosets of $T \cap H$ and consequently we can assume that $S \subseteq c(T \cap H)$ for some $c \in G$. Since also $S \subseteq bT$, we have $bT = cT$. Hence $\delta = \beta|_{c(T\cap H)}$ is an affine map satisfying $\delta|_S = u|_S$. Now $S \subseteq c(T \cap H)$ implies that $a = ch$ for some $h \in H$ and therefore

$$S = c\Big(H \backslash \bigcup_{j=1}^{m} hb_j K_j\Big) = c\Big((T \cap H) \backslash \bigcup_{j=1}^{m} hb_j K_j\Big).$$

If $hb_j K_j \cap (T \cap H) \neq \emptyset$, then $hb_j = tk$ for some $t \in (T \cap H)$ and $k \in K_j$ and hence

$$hb_j K_j \cap (T \cap H) = tK_j \cap (T \cap H) = t(K_j \cap T \cap H).$$

Thus, setting $A = T \cap H$ and $B_j = hb_j K_j \cap (T \cap H)$, we have

$$S = c\Big(A \backslash \bigcup_{j=1}^{m} B_j\Big),$$

where B_j is either empty or a coset in A. In addition, since K_j has infinite index in H and A has finite index in H, the subgroup corresponding to B_j has infinite index in A.

Since $u \in B(G)$ is uniformly continuous, the affine map $\delta : cA \to \mathbb{T}$ is uniformly continuous as well and hence extends to a continuous affine map $\overline{\delta} : c\overline{A} \to \mathbb{T}$.

Then $\overline{\delta}$ agrees with u on $\overline{S}$ since u is continuous. Let γ denote the continuous character of A associated with $\overline{\delta}$. Then $u(x) = \alpha\gamma(c^{-1}x)$ for all $x \in \overline{S}$.

Finally, since E_u is closed in G, E_u is a finite union of such sets $\overline{S}$ and on each such set $\overline{S}$, u is of the form stated in (2). This completes the proof of the theorem. $\square$

Theorem 9 above is due to Bert Schreiber for G abelian (TAMS 1970).

Corollary. *Let* u *be a power bounded element of* $A(G)$. *Then in the description of* E_u *and* $u|_{E_u}$ *in Theorem each* F_j *can be chosen to be a compact coset in* G.

Proof. We only have to note that E_u is compact and that every compact set in $\mathcal{R}(G)$ is a finite union of cosets of compact subgroups of G. □

Theorem 4 (Kaniuth, Lau and Ülger, JFA 2011)**.** *Let* G *be an arbitrary locally compact group and let* $u \in B(G)$ *be such that* E_u *is open in* G*. Then* u *is power bounded if and only if there exist*

(i) *pairwise disjoint open sets* $F_1, \ldots, F_n$ *in* $\mathcal{R}(G)$ *such that* $E_u = \bigcup_{j=1}^{n} F_j$ *and open subgroups* H_j *of* G *and* $a_j \in G$ *such that* $F_j \subseteq a_j H_j, \quad j = 1, \ldots, n,$ *and*

(ii) *characters* γ_j *of* H_j *and* $\alpha_j \in \mathbb{T}, \quad j = 1, \ldots, n,$ *such that*

$$u(x) = \alpha_j \gamma_j (a_j^{-1} x)$$

for all $x \in F_j$.

Let G be a discrete group and, for any subset E of G, let $C^*_\delta(E) = \overline{\langle \rho(x) : x \in E \rangle}$, the norm closure in $C^*_\rho(G)$ of the linear span of all operators $\rho(x)$, $x \in E$.

For any locally compact group G, let $C^*_\delta(G)$ denote the norm-closure in $\mathcal{B}\big(L^2(G)\big)$ of the linear span of all operators $\rho(x)$, $x \in G$.

Remark (Bekka, Kaniuth, Lau and Schlichting, Proc. A.M.S. 1996):

$C^*_\delta(G) \cong C^*_\rho(G_d) \Longleftrightarrow G$ contains an open subgroup H which is amenable as discrete.

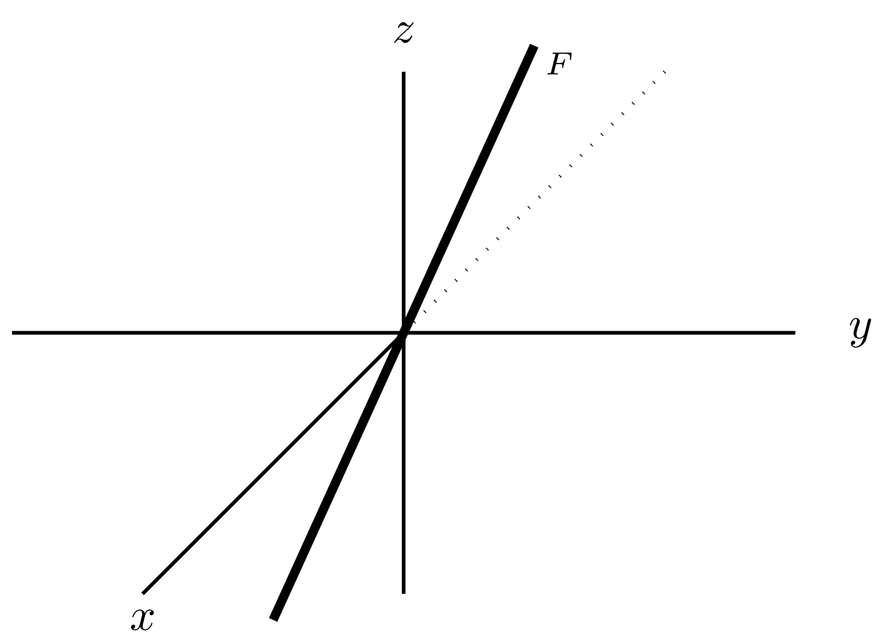
Theorem 5 (Kaniuth-Lau-Ulger, 2013)**.** *Let G be a locally compact group which contains an open subgroup H such that H_d is amenable and let $u \in B_\rho(G)$. Then u is power bounded if and only if* (i) *and* (ii) *hold.*

(i) $\|u\|_\infty \leq 1$ *and there exist pairwise disjoint sets $F_1, \ldots, F_n \in \mathcal{R}_c(G)$ such that $E_u = \cup_{j=1}^n F_j$, closed subgroups H_j of G and $a_j \in G$ such that $F_j \subseteq a_j H_j$, and characters γ_j of H_j and $\alpha_n \in \mathbb{T}$ such that $u(x) = \alpha_j \gamma_j (a_j^{-1} x)$ for all $x \in F_j$, $\quad 1 \leq j \leq n$.*

(ii) *For each $T \in C^*_\delta(G \backslash E_u)$, $\quad \langle u^n, T \rangle \to 0$ as $n \to \infty$.*

Geometric Form of Hahn-Banach Separation Theorem.

Every closed vector subspace of a locally convex space is the intersection of the closed hyperplanes containing it.



Lemma. *Let* H *be a closed subgroup of* G, *and* U *be a neighbourhood basis* $\mathcal{U}$ *of the identity of* G. *If* G *has the* H*-separation property,* *then*

$$(*) \qquad H = \bigcap_{U \in \mathcal{U}} \overline{HUH}.$$

Theorem (Kaniuth-Lau, 2003)**.** *If* G *is connected, then* G *has* H*-separation property* $\iff$ $(*)$ *holds.*

Open Problem 2: If G has property $(*)$ for each closed subgroup of G, does G have the invariant complementation property?

For **general** G

$$G-\ [\text{SIN}] \underset{\not\Leftarrow}{\Rightarrow} G \quad \text{has separation property} \Longrightarrow G \text{ has geometric separtion property}$$

$$\Downarrow \quad \not\Uparrow$$

Complementation property

For **connected** G :

$$G-\ [\text{SIN}] \Longleftrightarrow G \quad \text{has separation property} \Longleftrightarrow G \text{ has geometric separtion property}$$

5. Natural projections

Let A be a commutative Banach algebra with a BAI.

For $f \in A^*$ and $a \in A$, by $a \cdot f$ we denote the functional on A defined by $\langle a \cdot f, b \rangle = \langle f, ab \rangle$.

A projection $P : A^* \to A^*$ is said to be "*invariant*" (or A-invariant) if, for an $a \in A$ and $f \in A^*$, the equality $P(a \cdot f) = a \cdot P(f)$ holds. Similarly, a closed subspace X of A^* is said to be "invariant" if, for each $a \in A$ and $f \in X$, the functional $a \cdot f$ is in X (i.e. X is an A-module for the natural action $(a, f) \mapsto a \cdot f$). If there is an invariant projection from A^* onto a closed invariant subspace X of A^* then X is said to be "*invariantly complemented in* A^*".

We say that a projection $P : A^* \mapsto A^*$ is "*natural*" if, for each $\gamma \in \Delta(A)$, either $P(\gamma) = \gamma$ or $P(\gamma) = 0$.

If X is a closed invariant subspace of A^* and if there is natural projection P from A^* onto X we shall say that X is "*naturally complemented*" in A^*.

Lemma B. *Let $P : A^* \to A^*$ be a projection. Then*

a) *P is natural iff, for each $\gamma \in \Delta(A)$ and $a \in A$, $P(a \cdot \gamma) = a \cdot P(\gamma)$.*

b) *Every invariant projection $P : A^* \to A^*$ is natural.*

Theorem (Lau and Ulger, Trans. A.M.S. to appear)**.** *Let G be an amenable locally compact group, and I be a closed ideal in $A(G)$. Then $X = I^\perp$ is invariantly complemented $\Longleftrightarrow X$ is naturally complemented.*

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