Elementary t.d.l.c. second countable groups and applications

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- Countable discrete groups
- Compactly generated t.d.l.c. groups modulo a compact normal subgroup.

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- The compactly generated uniscalar t.d.l.c.s.c. group without compact open normal subgroup (Bhattacharjee, Macpherson)
- The non-discrete topologically simple t.d.l.c.s.c group with open abelian subgroup. (Willis)

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Theorem (Caprace, Monod)

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Remark

There is an ordinal rank on $\mathscr{E}$. Profinite and discrete groups are assigned rank zero.

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(iii) *If G is residually elementary, then $G \in \mathscr{E}$. In particular, $\mathscr{E}$ is closed under inverse limits.*

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- Induction on $rk(G)$ finishes the proof.

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(ii) There is a unique minimal closed normal subgroup whose quotient is elementary, denoted $Res_{\mathscr{E}}(G)$.

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If G is an elementary-free t.d.l.c.s.c. group, then $QZ(G) = \{1\}$ and the only locally normal abelian subgroup of G is $\{1\}$.

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So the series becomes:

$$\{1\} \leqslant Z(SL_3(\mathbb{Q}_p)) \leqslant SL_3(\mathbb{Q}_p) \leqslant GL_3(\mathbb{Q}_p)$$

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Can every t.d.l.c.s.c. group be “decomposed” into “basic” groups?

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We cannot omit the countable increasing union operation even for compactly generated groups. E.g. consider $\bigoplus_{i\in\mathbb{Z}}(PSL_3(\mathbb{Q}_p), U) \rtimes \mathbb{Z}$ where $\bigoplus_{i\in\mathbb{Z}}(PSL_3(\mathbb{Q}_p), U)$ is a local direct product.

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Answer

Yes. Indeed, for a slightly bigger class.

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(iii) *G/S is finite.*

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We adapt the following result of Caprace and Monod: in a compactly generated t.d.l.c. group G there is a compact $K \trianglelefteq G$ such that every filtering family of non-discrete closed normal subgroups of G/K has non-trivial intersection. Elementary-free is enough to show the desired adaptation to our setting.

Let $\mathcal{M}(G)$ denote the collection of minimal non-trivial normal subgroups given by **lemma (1).**

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This follows from **lemma (1)** since $Rad_{\mathscr{E}}$ and $Res_{\mathscr{E}}$ are characteristic subgroups.

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Fact ([2])

If G is a non-elementary topologically simple p-adic Lie group, then $G = S(\mathbb{Q}_p)^+$ for S an almost simple isotropic $\mathbb{Q}_p$-algebraic group.

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By the fact and results in algebraic group theory, $N_{min}(G) \simeq \prod_{N\in\mathcal{M}(G)} N$ and each $N \in \mathcal{M}(G)$ is compactly generated.

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- A similar argument gives G/S finite.

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The decomposition is thus

$$\{1\} \leqslant Z(SL_3(\mathbb{Q}_p)) \leqslant SL_3(\mathbb{Q}_p)$$

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(ii) The proof may generalize as it does not use much Lie theory. The current barrier is proving **lemma 1** in a more general setting.

Application 2: Surjectively universal groups

Definition

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Question (Gao)

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(iii) What sort of elementary groups appear as closed subgroups of $Aut(\mathcal{T}_d)$ with $\mathcal{T}_d$ the d-regular tree?

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(v) Is there an *SQ*-universal group for t.d.l.c.s.c groups? What about for elementary groups?

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