Multi-norms

H. G. Dales (Lancaster)

Fields Institute, Toronto

20/21 March 2014

References

BDP: O. Blasco, H. G. Dales, and H. L. Pham, *Equivalences involving* (p,q)*-multi-norms*, preprint.

DP1 : H. G. Dales and M. E. Polyakov, Homological properties of modules over group algebras, *PLMS* (3), 89 (2004), 390–426.

DP2 : H. G. Dales and M. E. Polyakov, Multi-normed spaces, *Dissertationes Math.*, 488 (2012), 1–165.

DDPR1 : H. G. Dales, M. Daws, H. L. Pham, and P. Ramsden, Multi-norms and injectivity of $L^p(G)$, *JLMS* (2), 86 (2012), 779–809.

DDPR2 : H. G. Dales, M. Daws, H. L. Pham, and P. Ramsden, Equivalence of multi-norms, *Dissertationes Math.*, 498 (2014), 1–53.

DLT: H. G. Dales, N. J. Laustsen, and V. Troitsky, *Multi-norms, quotients, and Banach lattices*, preliminary thoughts.

Basic definitions

Let $(E, \|\cdot\|)$ be a normed space. A **multi-norm** on $\{E^n : n \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_n)$ such that each $\|\cdot\|_n$ is a norm on E^n, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following hold for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in E$:

(A1) $\left\|(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\right\|_n = \|(x_1, \ldots, x_n)\|_n$ for each permutation σ of $\{1, \ldots, n\}$;

(A2) $\|(\alpha_1 x_1, \ldots, \alpha_n x_n)\|_n$

$$\leq (\max_{i \in \mathbb{N}_n} |\alpha_i|) \|(x_1, \ldots, x_n)\|_n$$

for each $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$;

(A3) $\|(x_1, \ldots, x_n, 0)\|_{n+1} = \|(x_1, \ldots, x_n)\|_n$;

(A4) $\|(x_1, \ldots, x_n, x_n)\|_{n+1} = \|(x_1, \ldots, x_n)\|_n$.

See [**DP2**].

Dual multi-norms

For a **dual multi-norm**, replace (A4) by:

(B4) $\|(x_1, \ldots, x_n, x_n)\|_{n+1} = \|(x_1, \ldots, x_{n-1}, 2x_n)\|_n$.

Let $(\|\cdot\|_n)$ be a multi-norm or dual multi-norm **based** on a space E. Then we have a **multi-normed space** and a **dual multi-normed space**, respectively. They are **multi-Banach spaces** and **dual multi-Banach spaces** when E is complete.

Let $\|\cdot\|_n$ be a norm on E^n. Then $\|\cdot\|_n'$ is the dual norm on $(E^n)'$, identified with $(E')^n$.

The **dual** of $(E^n, \|\cdot\|_n)$ is $((E')^n, \|\cdot\|_n')$. The dual of a multi-normed space is a dual multi-Banach space; the dual of a dual multi-normed space is a multi-Banach space.

What are multi-norms good for?

1) Solving specific questions - for example, characterizing when some modules over group algebras are injective [**DDPR1**]; see below.

2) Understanding the geometry of Banach spaces that goes beyond the shape of the unit ball.

3) Throwing some light on absolutely summing operators

4) Giving a theory [**DP2**] of 'multi-bounded linear operators' between Banach spaces. It gives a class of bounded linear operators that subsumes various known classes, and sometimes gives new classes.

5) Giving results about Banach lattices [**DP2**].

6) Giving a theory of decompositions [**DP2**] of Banach spaces generalizing known theories.

7) Giving a theory that 'is closed in the category'.

Conditions for modules to be injective

Let A be a Banach algebra. There is a condition for a Banach left A-module E to be 'injective'.

Let G be a locally compact group, and consider the following, which are all regarded as Banach left $L^1(G)$-modules in a natural way.

Theorem [**DP1**]

(1) $L^1(G)$ itself is injective iff G is discrete and amenable.

(2) $C_0(G)$ is injective iff G is finite.

(3) $L^\infty(G)$ is injective for all G.

(4) $M(G)$ is injective iff G is amenable. $\square$

What about $L^p(G)$ when $p > 1$?

An application

Let G be a locally compact group. The Banach space $L^p(G)$ is a Banach left $L^1(G)$-module in a canonical way.

Theorem - B. E. Johnson, 1972 Suppose that G is an amenable locally compact group and $1 < p < \infty$. Then $L^p(G)$ is an **injective** Banach left $L^1(G)$-module. $\square$

Long-standing conjecture The converse holds. Partial results in **DP, 2004**.

Theorem - DDPR1, 2012 Yes, G is amenable whenever $L^p(G)$ is injective for some (and hence all) $p \in (1, \infty)$. $\square$

This uses the theory of multi-norms. It gives various new, combinatorial characterizations of amenability.

A homework exercise

Let G be a group. Recall that G is **amenable** if, for each $\varepsilon > 0$ and each finite set F in G, there exists a finite set S in G such that

$$|Sx \Delta Sy| < \varepsilon\,|S| \quad (x, y \in F)\,.$$

This is **Folner's condition**.

We say that G is **pseudo-amenable** if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for each finite set F in G with $|F| \geq n_0$, there exists a finite set S in G such that

$$|SF| < \varepsilon\,|F|\,|S|\ .$$

Each amenable group is pseudo-amenable; a pseudo-amenable group cannot contain $\mathbb{F}_2$ as a subgroup.

Question Is every pseudo-amenable group already amenable?

Minimum and maximum multi-norms

Let $(E^n, \|\cdot\|_n)$ be a multi-normed space or a dual multi-normed space. Then

$$\max \|x_i\| \leq \|(x_1, \ldots, x_n)\|_n \leq \sum_{i=1}^{n} \|x_i\| \qquad (*)$$

for all $x_1, \ldots, x_n \in E$ and $n \in \mathbb{N}$.

Example 1 Set $\|(x_1, \ldots, x_n)\|_n^{\text{min}} = \max \|x_i\|$. This gives the **minimum** multi-norm.

Example 2 It follows from (*) that there is also a **maximum** multi-norm, which we call $(\|\cdot\|_n^{\text{max}} : n \in \mathbb{N})$.

Note that it is **not** true that $\sum_{i=1}^{n} \|x_i\|$ gives the maximum multi-norm — because it is not a multi-norm. (It is a dual multi-norm.)

A characterization of multi-norms

Give $\mathbb{M}_{m,n}$ a norm by identifying it with $\mathcal{B}(\ell_n^\infty, \ell_m^\infty)$.

Let E be a normed space. Then $\mathbb{M}_{m,n}$ acts from E^n to E^m in the obvious way.

Consider a sequence $(\|\cdot\|_n)$ such that each $\|\cdot\|_n$ is a norm on E^n and such that $\|x\|_1 = \|x\|$ for each $x \in E$.

Theorem This sequence of norms is a multi-norm if and only if

$$\|a \cdot \boldsymbol{x}\|_m \leq \|a : \ell_n^\infty \to \ell_m^\infty\| \, \|\boldsymbol{x}\|_n$$

for all $m, n \in \mathbb{N}$, $a \in \mathbb{M}_{m,n}$, and $\boldsymbol{x} \in E^n$. $\square$

Remark: We could calculate $\|a\|$ in different ways - for example, by identifying $\mathbb{M}_{m,n}$ with $\mathcal{B}(\ell_n^p, \ell_m^q)$ for other values of p and q. The case $p = q = 1$ gives a dual multi-norm. See **DLT** and the lecture of VT.

Another characterization

This is taken from [**DDPR1**]. It gives a 'coordinate-free' characterization.

Let $(E, \|\cdot\|)$ be a normed space. Then a c_0-**norm** on $c_0 \otimes E$ is a norm $\|\cdot\|$ such that:

1) $\|a \otimes x\| \leq \|a\| \, \|x\| \quad (a \in c_0, x \in E)$;

2) $T \otimes I_E$ is bounded on $(c_0 \otimes E, \|\cdot\|)$ with $\|T \otimes I_E\| = \|T\|$ whenever T is a compact operator on c_0;

3) $\|\delta_1 \otimes x\| = \|x\| \quad (x \in E)$.

Each c_0-norm is a reasonable cross-norm; we can replace 'T is a compact' by 'T is bounded'.

For the theory of tensor products, see the fine books of: J. Diestel, H. Jarchow, and A. Tonge; A. Defant and K. Floret; R. Ryan.

The connection

Theorem Multi-norms on $\{E^n : n \in \mathbb{N}\}$ correspond to c_0-norms on $c_0 \otimes E$.
The injective tensor product norm gives the minimum multi-norm, and the projective tensor product norm gives the maximum multi-norm $\square$

The recipe is: given a c_0-norm $\|\cdot\|$, set

$$\|(x_1, \ldots, x_n)\|_n = \left\| \sum_{j=1}^{n} \delta_j \otimes x_j \right\| \quad (x_1, \ldots, x_n \in E)\,.$$

Thus the theory of multi-norms could be a theory of norms on tensor products.

Banach lattices

Let $(E, \|\cdot\|)$ be a complex Banach lattice.

Then E is **monotonically bounded** if every increasing net in $E^+_{[1]}$ is bounded above, and **(Dedekind) complete** if every non-empty subset in E^+ which is bounded above has a supremum.

Examples $L^p(\Omega)$, $L^\infty(\Omega)$, or $C(K)$ with the usual norms and the obvious lattice operations are all Banach lattices.

Each Banach lattice L^p (for $p \in [1, \infty]$) and $C(K)$ (for K compact) is monotonically bounded, but c_0 is not monotonically bounded.

Each L^p-space is complete, but $C(K)$ is complete iff K is Stonean.

Banach lattice multi-norms

Let $(E, \|\cdot\|)$ be a complex Banach lattice.

Examples $L^p(\Omega)$, $L^\infty(\Omega)$, or $C(K)$ with the usual norms and the obvious lattice operations are all (complex) Banach lattices.

Definition [**DP2**] Let $(E, \|\cdot\|)$ be a Banach lattice. For $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$, set

$$\|(x_1, \dots, x_n)\|_n^L = \| \, |x_1| \vee \cdots \vee |x_n| \, \|$$

and

$$\|(x_1, \dots, x_n)\|_n^{DL} = \| \, |x_1| + \cdots + |x_n| \, \| \, .$$

Then $(E^n, \|\cdot\|_n^L)$ is a multi-Banach space. It is the **Banach lattice multi-norm**. Also $(E^n, \|\cdot\|_n^{DL})$ is a dual multi-Banach space. It is the **dual Banach lattice multi-norm**.

Each is the dual of the other.

A representation theorem

Clause (1) below is basically a theorem of **Pisier**, as given in a thesis of a student, **Marcolino Nhani**. There is an simplified proof in **DLT**. Clause (2) is a new dual version.

Theorem (**DLT**)

(1) Let $(E^n, \|\cdot\|_n)$ be a multi-Banach space. Then there is a Banach lattice X such that $(E^n, \|\cdot\|_n)$ is multi-isometric to $(Y^n, \|\cdot\|_n^L)$ for a closed subspace Y of X.

(2) Let $(E^n, \|\cdot\|_n)$ be a dual multi-Banach space. Then there is a Banach lattice X such that $(E^n, \|\cdot\|_n)$ is multi-isometric to $((X/Y)^n, \|\cdot\|_n^{DL})$ for a closed subspace Y of X. $\square$

Comparison with operator spaces

There is a huge industry connected with the theory of 'operator spaces'.

Definition Let E be a linear space, and consider an assignment of norms $\|\cdot\|_n$ on $\mathbb{M}_n(E)$ for each $n \in \mathbb{N}$; these norms are called the **matrix norms**. An **abstract operator space** on E is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ of matrix norms such that:

(M1) $\|\alpha v \beta\|_n \le \|\alpha\| \, \|v\|_m \, \|\beta\|$ for each $m, n \in \mathbb{N}$, $\alpha \in \mathbb{M}_{n,m}$, $\beta \in \mathbb{M}_{m,n}$, and $v \in \mathbb{M}_m(E)$.

(M2) $\|v \oplus w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}$ for each $m, n \in \mathbb{N}$, $v \in \mathbb{M}_m(E)$, and $w \in \mathbb{M}_n(E)$.

Ruan's theorem For each such system we can represent E as a closed subspace of $\mathcal{B}(H)$ for a Hilbert space H in such a way that the matrix norms are recovered in a canonical way. $\square$

This is an ℓ^2-theory; ours is an $\ell^1 - \ell^\infty$-theory.

An associated sequence

Let $(\|\cdot\|_n)$ be a multi-norm on $\{E^n : n \in \mathbb{N}\}$.

Define a **rate of growth** sequence via

$$\varphi_n(E) = \sup\{\|(x_1, \ldots, x_n)\|_n : \|x_i\| \leq 1\}\,.$$

Trivially, $1 \leq \varphi_n(E) \leq n$ for all $n \in \mathbb{N}$ and

$$\varphi_{m+n}(E) \leq \varphi_m(E) + \varphi_n(E)$$

for all $m, n \in \mathbb{N}$. What is the sequence $(\varphi_n(E))$?

In particular $(\varphi_n^{\max}(E))$ is the sequence associated with the maximum multi-norm.

It can be shown quite easily that $\varphi_n^{\max}(E)$ is

$$\sup\left\{\sum_{j=1}^{n}\left\|\lambda_j\right\|\right\},$$

where $\lambda_1, \ldots, \lambda_n \in E'$ and

$$\sum_{j=1}^{n}\left|\langle x,\, \lambda_j\rangle\right| \leq 1 \quad (x \in E_{[1]})\,.$$

Some examples

Theorem (i) For each $p \in [1, 2]$, we have

$$\varphi_n^{\max}(\ell_n^p) = \varphi_n^{\max}(\ell^p) = n^{1/p} \quad (n \in \mathbb{N}).$$

(ii) For each $p \in [2, \infty]$, there is a constant C_p such that

$$\sqrt{n} \le \varphi_n^{\max}(\ell_n^p) \le \varphi_n^{\max}(\ell^p) \le C_p\sqrt{n} \quad (n \in \mathbb{N}).$$

$\square$

[In general, I do not know the best constant C_p in the above inequality.]

Theorem Let E be an infinite-dimensional normed space. Then $\sqrt{n} \le \varphi_n^{\max}(E) \le n$ for each $n \in \mathbb{N}$.

Proof This uses Dvoretzky's theorem. $\square$

The Hilbert multi-norm

Let H be a Hilbert space. For each family $\mathbf{H} = \{H_1, \dots, H_n\}$ of closed subspaces of H such that $H = H_1 \perp \cdots \perp H_n$, set

$$r_{\mathbf{H}}((x_1, \dots, x_n)) = \left(\|P_1 x_1\|^2 + \cdots + \|P_n x_n\|^2\right)^{1/2},$$

where $P_i : H \to H_i$ for $i = 1, \dots, n$ is the projection, and then set

$$\|(x_1, \dots, x_n)\|_n^H = \sup_{\mathbf{H}} r_{\mathbf{H}}((x_1, \dots, x_n)).$$

Then we obtain a multi-norm $(\|\cdot\|_n^H : n \in \mathbb{N})$ based on H. It is the **Hilbert multi-norm**.

Summing norms - I

Let E be a normed space, and take $p \in [1, \infty)$. For $x_1, \ldots, x_n \in E$, set

$$\mu_{p,n}(x_1, \ldots, x_n) = \sup_{\lambda \in E'_{[1]}} \left\{ \left(\sum_{j=1}^{n} \left| \langle x_j, \lambda \rangle \right|^p \right)^{1/p} \right\} .$$

This is the **weak p-summing norm**. For example, we can see that

$$\mu_{1,n}(x_1, \ldots, x_n) = \sup \left\{ \left\| \sum_{j=1}^{n} \zeta_j x_j \right\| : \zeta_1, \ldots, \zeta_n \in \mathbb{T} \right\} .$$

For $\lambda_1, \ldots, \lambda_n \in E'$, we have

$$\mu_{1,n}(\lambda_1, \ldots, \lambda_n) = \sup \left\{ \sum_{j=1}^{n} \left| \langle x, \lambda_j \rangle \right| : x \in E_{[1]} \right\} .$$

Theorem [DP2] The dual of $\| \cdot \|_n^{\max}$ is $\mu_{1,n}$. $\square$

Summing norms - II

Again $1 \leq p \leq q < \infty$, and E and F are Banach spaces. For $T \in \mathcal{B}(E,F)$, $\pi_{q,p}^{(n)}(T)$ is

$$\sup\left\{\left(\sum_{j=1}^{n}\left\|Tx_j\right\|^q\right)^{1/q} : \mu_{p,n}(x_1,\ldots,x_n) \leq 1\right\}.$$

Definition Let $T \in \mathcal{B}(E,F)$. Suppose that

$$\pi_{q,p}(T) := \lim_{n\to\infty} \pi_{q,p}^{(n)}(T) < \infty\,.$$

Then T is (q,p)-**summing**; the set of these is $\Pi_{q,p}(E,F)$. This gives a Banach space.

We write $\pi_{q,p}^{(n)}(E)$ for $\pi_{q,p}^{(n)}(I_E)$ and $\pi_{q,p}(E)$ for $\pi_{q,p}(I_E)$. Also $\pi_p(E)$ for $\pi_{p,p}(E)$, etc.

In Memoriam: **Joram Lindenstrauss** (1936–2012) and **Aleksander Pełczyński** (1932–2012), founders of the theory of summing operators.

A connection

We write $\pi_{q,p}^{(n)}(E)$ for $\pi_{q,p}^{(n)}(I_E)$ and $\pi_{q,p}(E)$ for $\pi_{q,p}(I_E)$. Also $\pi_p(E)$ for $\pi_{p,p}(E)$, etc.

Theorem Let E be a normed space, and let $n \in \mathbb{N}$. Then

$$\varphi_n^{\max}(E) = \pi_1^{(n)}(E')\,.$$

If $E = F'$, then

$$\varphi_n^{\max}(E) = \pi_1^{(n)}(F)\,.$$

$\square$

The (p,q)−multi-norm

Let E be a Banach space, and take p, q with $1 \leq p \leq q < \infty$. Define

$$\|(x_1, \dots, x_n)\|_n^{(p,q)} = \sup \left\{ \left(\sum_{j=1}^{n} \left| \langle x_j, \lambda_j \rangle \right|^q \right)^{1/q} \right\},$$

taking the sup over all $\lambda_1, \dots, \lambda_n \in E'$ with $\mu_{p,n}(\lambda_1, \dots, \lambda_n) \leq 1$.

Fact: [**DP2**] $\{(E^n, \|\cdot\|_n^{(p,q)}) : n \in \mathbb{N}\}$ is a multi-Banach space.

Then $(\|\cdot\|_n^{(p,q)})$ is the (p,q)−**multi-norm** based on E.

Remarks (1) The $(1,1)$-multi-norm is the maximum multi-norm based on E.

(2) The (p,q)−multi-norm over E'', when restricted to E, is the (p,q)−multi-norm over E (by the principle of local reflexivity).

A connection

Let E be a normed space. Take $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$, and define

$$T_{\boldsymbol{x}} : (\zeta_1, \ldots, \zeta_n) \mapsto \sum_{j=1}^{n} \zeta_j x_j\,, \quad \mathbb{C}^n \to E\,.$$

Then $\mu_{p,n}(\boldsymbol{x}) = \left\| T_{\boldsymbol{x}} : \ell_n^{p'} \to E \right\|$ for $p \geq 1$.

It follows that

$$\|\boldsymbol{x}\|_n^{(p,q)} = \pi_{q,p}(T'_{\boldsymbol{x}} : E' \to c_0)\,.$$

This leads to:

Theorem Let E be a normed space, and suppose that $1 \leq p \leq q < \infty$. Then the (p, q)-multi-norm induces the norm on $c_0 \otimes E$ given by embedding $c_0 \otimes E$ into $\Pi_{q,p}(E', c_0)$. $\square$

The (p, p)–multi-norm

For Banach spaces E and F, the (right) **Chevet–Saphar norm** d_p on $E \otimes F$ is defined as

$$d_p(z) = \inf \left\{ \mu_{p',n}(x_1, \ldots, x_n) \left(\sum_{i=1}^{n} \|y_i\|^p \right)^{1/p} \right\},$$

taking the inf over $\{z = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F\}$. This norm is what is called a **uniform cross-norm**.

Theorem Let E be a normed space. Then the (p, p)-multi-norm (regarded as a norm on $c_0 \otimes E$) is the Chevet–Saphar norm d_p.

Proof The (p, p)-multi-norm comes from the embedding of $c_0 \otimes E$ into $\Pi_p(E', c_0)$. The latter agrees isometrically with the class of p-integral maps from E' into c_0 - and the p-integral norm is the norm of the induced functional on

$$E' \widehat{\otimes}_{g_p'} \ell^1 = \ell^1 \widehat{\otimes}_{d_p'} E'.$$

We use the facts that c_0 has MAP and d_p is an accessible tensor norm. $\square$

A question

Question But what if we go to the $(p, q)-$multi-norm? What tensor product does it explicitly correspond to? How do we calculate dual spaces?

(p, q)-**invariant means**

Let G be a locally compact group, and take p, q with $1 \leq p \leq q < \infty$. A mean $\Lambda \in L^\infty(G)'$ is (p, q)-**invariant** if the set $\{s \cdot \Lambda : s \in G\}$ is (p, q)-multi-bounded (see below). The group G is (p, q)-**amenable** if there is such a mean.

Key Theorem [**DDPR1**] In fact, G is (p, q)-amenable if and only if it is amenable (and several other characterizations).

Proof This uses characterizations of weak compactness, the Krein–Smulyan theorem, and the Ryll-Nardzewski fixed point theorem. $\square$

Concave multi-norms

Let E be a Banach lattice, and take p, q with $1 \leq p \leq q < \infty$.

Definition The $[p,q]$-**concave multi-norm** is given by

$$\|\boldsymbol{x}\|_n^{[p,q]} = \sup\left\{\left(\sum_{j=1}^{n}\left|\langle x_j,\, \lambda_j\rangle\right|^q\right)^{1/q}\right\},$$

where the supremum is taken over all those $\lambda_1, \dots, \lambda_n \in E'$ such that

$$\left\|\left(\sum_{j=1}^{n}\left|\lambda_j\right|^p\right)^{1/p}\right\| \leq 1\,.$$

(The relevant term is defined by the Krivine calculus.)

Theorem The sequence $(\|\cdot\|_n^{[p,q]})$ is a multi-norm. $\square$

Concave operators

The above $[p,q]$-multi-norms multi-norms are related to the '(q,p)-concave operators between Banach lattices' in the same way as (p,q)-multi-norms are related to (q,p)-summing operators. Thus we can use some theorems of Maurey.

Proposition Let E be a Banach lattice. Then:

(i) for $1 \leq p_1 \leq q_1 < \infty$ and $1 \leq p_2 \leq q_2 < \infty$, we have $(\|\cdot\|_n^{[p_2,q_2]}) \leq (\|\cdot\|_n^{[p_1,q_1]})$ whenever both $1/p_1 - 1/q_1 \leq 1/p_2 - 1/q_2$ and $q_1 \leq q_2$;

(ii) for $1 \leq p \leq q < \infty$, $(\|\cdot\|_n^{[p,q]}) \leq (\|\cdot\|_n^{(p,q)})$;

(iii) for $1 \leq p < q < \infty$, $(\|\cdot\|_n^{[p,q]}) \cong (\|\cdot\|_n^{[1,q]})$;

(iv) for $q > 2$, we have $(\|\cdot\|_n^{[1,q]}) \cong (\|\cdot\|_n^{(1,q)})$;

(v) $(\|\cdot\|_n^{(1,2)}) \preccurlyeq (\|\cdot\|_n^{[2,2]})$. $\square$

The standard t-multi-norm on $L^r(\Omega)$

Let Ω be a measure space, and take r, t with $1 \le r \le t < \infty$. We consider the Banach space $L^r(\Omega)$ (e.g., ℓ^r), with the usual L^r-norm $\|\cdot\|$.

For each family $\mathbf{X} = \{X_1, \dots, X_n\}$ of pairwise-disjoint measurable subsets of Ω such that $X_1 \cup \cdots \cup X_n = \Omega$, we set

$$r_{\mathbf{X}}((f_1, \dots, f_n)) = \left(\left\|P_{X_1} f_1\right\|^t + \cdots + \left\|P_{X_n} f_n\right\|^t\right)^{1/t},$$

where $P_X : L^r(\Omega) \to L^r(X)$ is the natural projection.

Finally, $\|(f_1, \dots, f_n)\|_n^{[t]} = \sup_{\mathbf{X}} r_{\mathbf{X}}((f_1, \dots, f_n))$.

Then $(\|\cdot\|_n^{[t]})$ is the **standard t-multi-norm** (on $L^r(\Omega)$) from [**DP2**].

Remark Suppose that $t = r$. Then

$$\|(f_1, \dots, f_n)\|_n^{[r]} = \|\, |f_1| \vee \cdots \vee |f_n| \,\| ,$$

and so $(\|\cdot\|_n^{[t]})$ is equal to the lattice multi-norm on $L^r(\Omega)$.

Concave and standard multi-norms

Theorem Suppose that $1 \leq r \leq t < \infty$, and set $1/v = 1/r - 1/t$. Then the standard t-multi-norm is equal to the $[1, v']$-concave multi-norm on ℓ^r.

In particular, the Banach lattice multi-norm on ℓ^r is the $[1, 1]$-concave multi-norm on ℓ^r. $\square$

Multi-convergence

Definition Let $(E^n, \|\cdot\|_n)$ be a multi-normed space. Then a sequence (x_i) is **multi-null**, written

$$\operatorname*{Lim}_i x_i = 0$$

if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|(x_{n_1}, \ldots, x_{n_k})\|_k < \varepsilon \quad (n_1, \ldots, n_k \geq n_0,\, k \in \mathbb{N})\,.$$

Example Let $(E, \|\cdot\|)$ be an 'order-continuous' Banach lattice, and consider the Banach lattice multi-norm on $\{E^n : n \in \mathbb{N}\}$. Then a sequence is a multi-null sequence if and only if it converges to 0 'in order'. □

Definition An operator is **multi-continuous** if it takes multi-null sequences to multi-null sequences.

Multi-bounded sets and operators

Let $(E^n, \|\cdot\|_n)$ be a multi-normed space. A subset B of E is **multi-bounded** if

$$c_B := \sup_{n\in\mathbb{N}} \{\|(x_1, \dots, x_n)\|_n : x_1, \dots, x_n \in B\} < \infty\,.$$

Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be multi-Banach spaces. An operator $T \in \mathcal{B}(E, F)$ is **multi-bounded** if $T(B)$ is multi-bounded in F whenever B is multi-bounded in E. The set of these is a linear subspace $\mathcal{M}(E, F)$ of $\mathcal{B}(E, F)$; $\mathcal{M}(E)$ is a Banach algebra.

Theorem An operator $T \in \mathcal{B}(E, F)$ is multi-bounded iff it is multi-continuous. $\square$

For $T_1, \dots, T_n \in \mathcal{M}(E, F)$, set

$$\|(T_1, \dots, T_n)\|_{mb,n} = \sup\{c_{T_1(B)\cup\cdots\cup T_n(B)} : c_B \leq 1\}\,.$$

Theorem Now $(\mathcal{M}(E, F)^n, \|\cdot\|_{mb,n})$ is a multi-Banach space, and $(\mathcal{M}(E)^n, \|\cdot\|_{mb,n})$ is a ‘multi-Banach algebra’. $\square$

Examples of $\mathcal{M}(E,F)$

Theorem Always

$$\mathcal{N}(E,F) \subset \mathcal{M}(E,F) \subset \mathcal{B}(E,F)\,. \qquad \square$$

Theorem We can have $\mathcal{M}(E,F) = \mathcal{B}(E,F)$ and $\mathcal{M}(F,E) = \mathcal{N}(F,E)$. So there is no 'multi-Banach isomorphism theorem'. $\square$

Theorem We can have $\mathcal{K}(E) \not\subset \mathcal{M}(E)$. $\square$

Multi-bounded maps between Banach lattices

Theorem Let E and F be Banach lattices, and define $\mathcal{M}(E,F)$ with respect to the lattice multi-norms on E and F.

(i) Suppose that F is monotonically bounded. Then $\mathcal{M}(E,F) = \mathcal{B}_b(E,F)$.

(ii) Suppose, further that F has the Nakano property. Then, further,

$$\|T\|_{mb} = \|T\|_b \quad (T \in \mathcal{B}_b(E,F))\,.$$

(iii) Suppose that F is monotonically bounded and Dedekind complete. Then

$$\mathcal{M}(E,F) = \mathcal{B}_r(E,F) = \mathcal{B}_b(E,F)\,,$$

and $\|\cdot\|_{mb}$ and $\|\cdot\|_r$ are equivalent on $\mathcal{B}_r(E,F)$. $\square$

Questions about multi-norms on Banach lattices

Question What are the subsets B of ℓ^r that are (p, q)-multi-bounded? Which operators between these spaces are multi-bounded - when we put maybe different (p, q)-multi-norms on maybe different ℓ^r spaces?

Question What happens when 'suppose' does not apply in the previous slide?

Do any of these questions lead to interesting classes of operators?

Equivalences of multi-norms

Definition [DP2] Let $(E, \|\cdot\|)$ be a normed space. Suppose that both $(\|\cdot\|_n^1)$ and $(\|\cdot\|_n^2)$ are multi-norms on E. Then $(\|\cdot\|_n^2)$ **dominates** $(\|\cdot\|_n^1)$, written $(\|\cdot\|_n^1) \preccurlyeq (\|\cdot\|_n^2)$, if there is a constant $C > 0$ such that

$$\|\boldsymbol{x}\|_n^1 \leq C\,\|\boldsymbol{x}\|_n^2 \quad (\boldsymbol{x} \in E^n,\, n \in \mathbb{N})\,.$$

The two multi-norms are **equivalent**, written

$$(\|\cdot\|_n^1) \cong (\|\cdot\|_n^2)$$

if each dominates the other.

We wish to decide when various pairs of multi-norms are mutually equivalent - for example, what about (p, q)-multi-norms on ℓ^r?

Clearly equivalent multi-norms have equivalent rates of growth (via the sequences (φ_n)), but the converse does not hold.

Equivalences of the Hilbert multi-norm

Theorem [**DDPR2**] Let H be an infinite-dimensional (complex) Hilbert space. Then:

(i) the Hilbert and $(2,2)$-multi-norms are equal;

(ii) $\|\cdot\|_n^H \leq \|\cdot\|_n^{\max} \leq \frac{2}{\sqrt{\pi}} \|\cdot\|_n^H$ for all $n \in \mathbb{N}$ (and the constant is best-possible);

(iii) the above norms are also equivalent to the (p,p)-multi-norm whenever $p \in [1,2]$, but they are not equivalent to any (p,q)-multi-norm for which $p < q$.

(iv) but the (p,p)- and (q,q)- multi-norms are not equivalent when $p \neq q$ and $\max\{p,q\} > 2$. $\square$

Interpretation in terms of summing operators

Theorem (**DDPR2**) Let E be a normed space. Then

$$(\|\cdot\|_n^{(p_1,q_1)}) \cong (\|\cdot\|_n^{(p_2,q_2)})$$

if and only if

$$\Pi_{q_1,p_1}(E',F) = \Pi_{q_2,p_2}(E',F)$$

as subsets of $\mathcal{B}(E',F)$ for each Banach space F. $\square$

Thus the theory of the equivalence of multi-norms could be a theory of (q,p)-summing operators.

Some curves

Look at the 'triangle'

$$\mathcal{T} = \{(p,q) : 1 \leq p \leq q < \infty\}\,.$$

For $c \in [0,1)$, look at the curve $\mathcal{C}_c$:

$$\mathcal{C}_c = \left\{(p,q) \in \mathcal{T} : \frac{1}{p} - \frac{1}{q} = c\right\}\,.$$

Take $r \in (1,\infty)$. Then the curve $\mathcal{C}_{1/r}$ meets the line $p = 1$ at the point $(1, r')$. The union of these curves is $\mathcal{T}$.

Two points $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ in $\mathcal{T}$ are **equivalent for a normed space** E if the corresponding multi-norms $(\|\cdot\|_n^{(p_1,q_1)})$ and $(\|\cdot\|_n^{(p_2,q_2)})$ based on E are equivalent.

First main question: When are two points in $\mathcal{T}$ equivalent for ℓ^r (where $r \geq 1$)?

First result

The following is a fairly easy result from the theory of absolutely summing operators.

Theorem Let E be a normed space, and suppose that

$$1 \leq p_1 \leq q_1 < \infty \quad \text{and} \quad 1 \leq p_2 \leq q_2 < \infty\,.$$

Then $(\|\cdot\|_n^{(p_2,q_2)}) \leq (\|\cdot\|_n^{(p_1,q_1)})$ whenever both $1/p_1 - 1/q_1 \leq 1/p_2 - 1/q_2$ and $q_1 \leq q_2$. $\square$

Picture 1: The (p,q)-triangle

Picture 2: Larger/smaller (p,q)-multi-norms

A calculation

The following calculation gives us a start. It will show non-equivalence between some (p, q)-multi-norms.

We calculate $\|(\delta_1, \ldots, \delta_n)\|_n^{(p,q)}$ acting on ℓ^r (for $r \geq 1$ and $1 \leq p \leq q < \infty$). The answer is:

$$\left\{ \begin{array}{cl} n^{1/r+1/q-1/p} & \text{when } p < r \text{ and } 1/p - 1/q \leq 1/r, \\ \\ 1 & \text{when } 1/p - 1/q > 1/r, \\ \\ n^{1/q} & \text{when } p \geq r\,. \end{array} \right\}$$

There are similar calculations involving $\|(f_1, \ldots, f_n)\|_n^{(p,q)}$, where

$$f_i = \frac{1}{n^{1/r}}(\zeta^{-i}, \zeta^{-2i}, \ldots, \zeta^{-ni}, 0, 0, \ldots)$$

and $\zeta = \exp(2\pi \mathrm{i}/n)$.

Some tools

The **generalized Hölder's inequality** gives us:

Lemma Take p, q_1, q_2 with $1 \le p \le q_1 < q_2$. Then, for $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, the number $\|\boldsymbol{x}\|_n^{(p,q_2)}$ is equal to

$$\sup \left\{ \|(\zeta_1 x_1, \dots, \zeta_n x_n)\|_n^{(p,q_1)} : \sum_{j=1}^{n} \left|\zeta_j\right|^u \le 1 \right\},$$

where u satisfies $1/u = 1/q_1 - 1/q_2$. □

Theorem (**Khintchine's inequality**): for each $u > 0$, there exist constants A_u and B_u such that

$$\begin{aligned} A_u \left(\sum_{j=1}^{n} \left|\alpha_j\right|^2 \right)^{1/2} &\le \left(\int_0^1 \left| \sum_{j=1}^{n} \alpha_j r_j(t) \right|^u \mathrm{d}t \right)^{1/u} \\ &\le B_u \left(\sum_{j=1}^{n} \left|\alpha_j\right|^2 \right)^{1/2} \end{aligned}$$

for all $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and all $n \in \mathbb{N}$. Here the r_j are the **Rademacher functions**. □

A factorization theorem

We use the following factorization theorem of **Grothendieck**.

Lemma Let $F = L^s(\Omega)$, where Ω is a measure space and $s \geq 1$. Take $u > s$ and $u = 2$ in the cases where $s > 2$ and $s \in [1, 2]$, respectively. Then there is a constant $K > 0$ such that, for each $n \in \mathbb{N}$ and each $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in F^n$ with $\mu_{1,n}(\boldsymbol{\lambda}) = 1$, there exist $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$ and $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n) \in F^n$ such that:

(i) $\lambda_j = \zeta_j \nu_j \ (j \in \mathbb{N}_n)$;

(ii) $\sum_{j=1}^n \left|\zeta_j\right|^u \leq 1$;

(iii) $\mu_{u',n}(\boldsymbol{\nu}) \leq K$.

In the key case where $s \in [1, 2]$, we can take $K = K_G$, which is Grothendieck's constant. $\square$

The case where $r = 1$

Take two (p, q)–multi-norms based on ℓ^1, say $(\|\cdot\|_n^{(p_1,q_1)})$ and $(\|\cdot\|_n^{(p_2,q_2)})$. The above calculation shows that a necessary condition for equivalence is that $q_1 = q_2 = q$, say.

Now $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{(1,q)})$ whenever $1 \le p < q$, but they are not equivalent to $(\|\cdot\|_n^{(q,q)})$.

The latter depends on an example of Stephen Montgomery-Smith (Thesis, Cambridge, 1988):

Let I_n be the identity map from ℓ_n^∞ to the Lorentz space $\ell_n^{q,1}$. Then

$$\pi_{q,q}(I_n) \sim n^{1/q}(1+\log n)^{1-1/q}, \quad \pi_{q,1}(I_n) \sim n^{1/q}.$$

Now for the case where $r > 1$.

The minimum multi-norm

Theorem [**BDP**] Let E be a Banach space with type $u \in [1, 2]$, and take $s \in [1, u]$. Then there is a constant $K > 0$ such that

$$\|\boldsymbol{x}\|_n^{(1,s')} \leq K \|\boldsymbol{x}\|_n^{\min} \quad (\boldsymbol{x} \in E^n,\, n \in \mathbb{N}) \,. \qquad \square$$

Recall that a normed space E has **type** u for $1 \leq u \leq 2$ if there is a constant $K \geq 0$ such that

$$\left(\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^2 \mathrm{d}t \right)^{1/2} \leq K \left(\sum_{j=1}^n \left\| x_j \right\|^u \right)^{1/u}$$

The space $L^r(\Omega)$ has type $\min\{r, 2\}$.

Full solution for $r \geq 2$

Theorem (**BDP**) Take $r \geq 2$ and $E = \ell^r$. Then the triangle $\mathcal{T}$ decomposes into the following (mutually disjoint) equivalence classes:

- $\mathcal{T}_{\mathrm{min}} := A_r = \{(p,q) \in \mathcal{T} \colon 1/p - 1/q \geq 1/2\}$;

- the curves $\mathcal{T}_c := \{(p,q) \in \mathcal{C}_c \colon 1 \leq p \leq 2\}$, for $c \in [0, 1/2)$;

- the singletons $\mathcal{T}_{(p,q)} := \{(p,q)\}$ for $(p,q) \in \mathcal{T}$ with $p > 2$.

Picture 3: Equivalence classes when $r \geq 2$.

Sketch of proof

To show that alleged disjoint classes are indeed disjoint use the elementary exercises where one can to separate out classes; this does not seem to work when $p_1 \geq r$ and $p_2 > r$, and, in this case, we must use the deeper results involving Schatten classes, coupled with Khintchine's inequality and the 'Orlicz property'.

To show that we do have equivalence where claimed, use the previous lemmas on minimum multi-norms and on curves.

The case where $1 < r < 2$

Picture 4: Equivalence classes when $1 < r < 2$.

Open cases

There are open cases only when $1 < r < 2$.

First open case Does

$$\Pi_{(q,r)}(\ell^{r'}, c_0) = \Pi_{(q,2q/(q-2))}(\ell^{r'}, c_0)$$

when $r < 2$ and $q = 2r/(2-r)$? That is: 'Do we have equivalence on the flat bit?' No idea.

Second open case Consider the points on the curve $\mathcal{C}_c$ with $1 \le p \le r$; the left-hand point of this curve is $(1, 1/(1-c))$, and each such point with $1 \le p < r$ is equivalent to it.

This leaves open the question whether the point (r, u_c) is equivalent to $(1, 1/(1-c))$. An old example of Kwapień shows that this is not the case for $c = 0$, and it is proved in BDP that it is true for $c \in (1/2, 1/r)$, but we do not know what happens when $c \in (0, 1/2]$.

This should be re-solvable.

(p,q)-multi-norms and standard multi-norms

Fix the space ℓ^r, where $r \geq 1$, and fix $t \geq r$, so the standard t-multi-norm on ℓ^r is defined.

We wish to determine

$$B_{r,t} := \left\{ (p,q) \in \mathcal{T} : (\|\cdot\|_n^{[t]}) \preccurlyeq (\|\cdot\|_n^{(p,q)}) \right\}$$

and

$$D_{r,t} := \left\{ (p,q) \in \mathcal{T} : (\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[t]}) \right\} .$$

Fact There is no (p,q)-multi-norm which is equivalent to the standard t-multi-norm on ℓ^r if and only if these regions are disjoint.

Conjecture from DDPR2 This is always the case whenever $r > 1$.

An easy first step

Theorem Fix $r \geq 1$. Then

$$B_{r,t} = \{(p,q) \in \mathcal{T} : 1/p - 1/q \leq 1/r - 1/t,\ q \leq t\}\,.$$

Reason It is easy to see that we always have $(\|\cdot\|_n^{[t]}) \leq (\|\cdot\|_n^{(r,t)})$, and so this follows from earlier diagrams. $\square$

Picture 5: The set $B_{r,t}$.

The case where $r = 1$

Theorem Take $t > 1$. Then

$$D_{1,t} = \{(p,q) \in \mathcal{T} : q \geq \max\{t,p\}\} \setminus \{(t,t)\}\,,$$

whereas

$$B_{1,t} = \{(p,q) \in \mathcal{T} : q \leq t\}\,.$$

Hence $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{[t]})$ on the space ℓ^1 if and only if $p = q = t = 1$ or $p < q = t$.

Picture 6: The sets $B_{1,t}$ and $D_{1,t}$.

Proof Most of this follows from the exercises, save for the fact that

$$(\|\cdot\|_n^{(p,p)}) \preccurlyeq (\|\cdot\|_n^{(1,t)}) = (\|\cdot\|_n^{[t]})$$

when $q = p > t$. This follows from a result of **Pisier** that says that $\Pi_{1,t}(\ell^\infty) \subset \Pi_p(\ell^\infty)$ in this case. $\square$

The case where $r \geq 2$

This is also rather easy; it follows from earlier calculations.

Theorem Take $t \geq r \geq 2$. Then

$$D_{r,t} = \{(p,q) \in \mathcal{T} : 1/p - 1/q \geq 1/2\}\,,$$

whereas

$$B_{r,t} = \{(p,q) \in \mathcal{T} : 1/p - 1/q \leq 1/r - 1/t,\ q \leq t\}\,.$$

Thus $D_{r,t}$ and $B_{r,t}$ are indeed disjoint. $\square$

Picture 7: The sets $B_{r,t}$ and $D_{r,t}$ for $r \geq 2$.

The case where $1 < r < 2$

This seems much harder and more interesting.

By a rather deep calculation we have:

Theorem Take $t \geq r > 1$, and consider the space ℓ^r. Set $1/s = 1/r - 1/t$. For example, when $s \geq 2$, then

$$D_{r,t} = \left\{ (p,q) : \frac{1}{p} - \frac{1}{q} \geq \frac{1}{s} \right\},$$

which is again disjoint from $B_{r,t}$. $\square$

Only partially solved: the case where $t \geq r > 1$ and $r < 2$ and $1/r - 1/t > 1/2$.

Counter to the conjecture

Here, if you look carefully, the two sets do (just) overlap.

Theorem Suppose that $1 < r < 2$, that $t \geq r$, and that $1 \leq p \leq q < \infty$, and consider the space $\ell^{\,r}$. Suppose further that $1/r - 1/t > 1/2$. Then $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{[t]})$ whenever

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{r} - \frac{1}{t} \quad \text{and} \quad 1 \leq p \leq r\,.$$

$\square$

A question concerning matrices

It is surprising that we can 'reduce' the calculation of $D_{r,r}$ to one about matrices.

Given a matrix $A = (a_{i,j})$, we form $|A|$ by replacing each $a_{i,j}$ by $\left|a_{i,j}\right|$.

Theorem Take $r \geq 1$. Then the following conditions on a point $(p,q) \in \mathcal{T}$ are equivalent:

(a) $(\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[r]})$ on ℓ^r;

(b) there exists a constant $C > 0$ such that

$$\|\,|A| : \ell_m^r \to \ell_n^q\| \leq C\,\|A : \ell_m^r \to \ell_n^p\|$$

for every $m,n \in \mathbb{N}$ and every $n \times m$ matrix A;

(c) $|T| \in \mathcal{B}(\ell^r, \ell^q)$ whenever $T \in \mathcal{B}(\ell^r, \ell^p)$. $\square$

Thus our result gives a result about matrices that might possibly be new.

Theorem Take $r > 1$ and $1 \leq p \leq q < \infty$. Then there exists a constant $C > 0$ such that

$$\| \, |A| : \ell_m^r \to \ell_n^q \| \leq C \, \|A : \ell_m^r \to \ell_n^p\|$$

for every $m, n \in \mathbb{N}$ and every $n \times m$ matrix A if and only if $1/p - 1/q \geq 1/2$.

References

BDP: O. Blasco, H. G. Dales, and H. L. Pham, *Equivalences involving* (p, q)*-multi-norms*, in preparation.

DP1 : H. G. Dales and M. E. Polyakov, Homological properties of modules over group algebras, *PLMS* (3), 89 (2004), 390–426.

DP2 : H. G. Dales and M. E. Polyakov, Multi-normed spaces, *Dissertationes Math.*, 488 (2012), 1–165.

DDPR1 : H. G. Dales, M. Daws, H. L. Pham, and P. Ramsden, Multi-norms and injectivity of $L^p(G)$, *JLMS* (2), 86 (2012), 779–809.

DDPR2 : H. G. Dales, M. Daws, H. L. Pham, and P. Ramsden, Equivalence of multi-norms, *Dissertationes Math.*, 498 (2014), 1–53.

DLT: H. G. Dales, N. J. Laustsen, and V. Troitsky, *Multi-norms, quotients, and Banach lattices*, preliminary thoughts.