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Arbitrages in a progressive enlargement of filtration

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We consider a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ and a random time τ (i.e., a **positive finite** $\mathcal{A}$-measurable random variable).

We assume that the financial market where a risky asset with price S (an $\mathbb{F}$-adapted positive process) and a riskless asset $S^0 \equiv 1$ are traded is arbitrage free.

More precisely, we assume w.l.g. that S is a $(\mathbb{P}, \mathbb{F})$ (local) martingale.

We denote by $\mathbb{G}$ the progressively enlarged filtration of $\mathbb{F}$ by τ, i.e.,

$$\mathcal{G}_t = \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge (t+\epsilon))$$

Our aim is to determine if, using $\mathbb{G}$-predictable strategies, one can produce arbitrages.

In the particular case where τ is an $\mathbb{F}$-stopping time, the enlarged filtration and the reference filtration are the same. In that case, there are no arbitrage opportunities in the enlarged filtration.

Arbitrages

Let $\mathbb{K}$ be one of the filtrations $\{\mathbb{F}, \mathbb{G}\}$.

For $a \in \mathbb{R}_+$, an element $\theta \in L^{\mathbb{K}}(S)$ is said to be an **a-admissible $\mathbb{K}$-strategy** if $(\theta \centerdot S)_\infty := \lim_{t\to\infty} (\theta \centerdot S)_t$ exists and $V_t(0,\theta) := (\theta \centerdot S)_t \geq -a$ $\mathbb{P}$-a.s. for all $t \geq 0$.

We denote by $\mathcal{A}_a^{\mathbb{K}}$ the set of all a-admissible $\mathbb{K}$-strategies. A process $\theta \in L^{\mathbb{K}}(S)$ is called an *admissible* $\mathbb{K}$*-strategy* if $\theta \in \mathcal{A}^{\mathbb{K}} := \bigcup_{a\in\mathbb{R}_+} \mathcal{A}_a^{\mathbb{K}}$.

An admissible strategy yields an **Arbitrage Opportunity** if $V(0,\theta)_\infty \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}\big(V(0,\theta)_\infty > 0\big) > 0$. In order to avoid confusions, we shall call these arbitrages ***classical arbitrages***.

If there exists no such $\theta \in \mathcal{A}^{\mathbb{K}}$ we say that the financial market $\mathcal{M}(\mathbb{K}) := (\Omega, \mathbb{K}, \mathbb{P}; S)$ satisfies the No Arbitrage **(NA)** condition.

No Free lunch with Vanishing Risk (NFLVR) holds in the financial market $\mathcal{M}(\mathbb{K})$ if and only if there exists an Equivalent Martingale Measure in $\mathbb{K}$, i.e. $\mathbb{Q} \sim \mathbb{P}$ so that the process S is a $(\mathbb{Q}, \mathbb{K})$-local martingale. If NFLVR holds, there are no classical arbitrages.

Enlargement of filtration results

We define the right-continuous with left limits $\mathbb{F}$-supermartingale

$$Z_t := \mathbb{P}\left(\tau > t \mid \mathcal{F}_t\right).$$

One can write

$$Z = m - A^o$$

where m is an $\mathbb{F}$-martingale and A^o is the $\mathbb{F}$-dual optional projection (an increasing process) of $A = \mathbb{1}_{[\![\tau,\infty[\![}$.

Note that m is non-negative: indeed $m_t = \mathbb{E}(A^o_\infty|\mathcal{F}_t)$.

The $\mathbb{F}$-supermartingale

$$\widetilde{Z}_t := \mathbb{P}\left(\tau \geq t \;\middle|\; \mathcal{F}_t\right)$$

will play a particular rôle in the following. One has $\tilde{Z} = Z + \Delta A^o$, hence the supermartingale $\widetilde{Z}$ admits a decomposition as

$$\widetilde{Z} = m - A^o_- \,.$$

Note that Z_- and $\widetilde{Z}$ do not vanish on $[0, \tau]$.

An important obvious remark

Assume that the financial market where (S^0, S) are traded is complete.

If $m_\tau \geq 1$ and $\mathbb{P}(m_\tau > 1) > 0$, then, there are arbitrages before τ.

Due to the completion hypothesis, the positive martingale m satisfies $m_t - 1 = \int_0^t \varphi_s dS_s$, hence φ is an admissible self-financing strategy, therefore φ corresponds to a classical arbitrage.

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The completeness of the $\mathbb{F}$ market seems to be an essential hypothesis to have classical arbitrages:

Let W^1, W^2 be a standard 2-dimensional Brownian motion and

$$dS_t = S_t f(W_t^2) dW_t^1$$

Under regularity assumptions $\mathbb{F}^S = \mathbb{F}^1 \vee \mathbb{F}^2$. Let τ be an $\mathbb{F}^2$ honest time (hence an $\mathbb{F}^S$ honest time). Since W^1 is an $\mathbb{F}^1 \vee \sigma(\tau \wedge \cdot)$ martingale, there are no arbitrages in the enlarged filtration.

Some particular cases

Density hypothesis

If there exists a positive $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(\omega, u) \to \alpha_t(\omega, u)$ which satisfies for any Borel bounded function φ,

$$\mathbb{E}(\varphi(\tau)|\mathcal{F}_t) = \int_{\mathbb{R}_+} \varphi(u)\alpha_t(u)\nu(du), \quad \mathbb{P} - a.s.$$

where ν is the law of τ, then NFLVR holds for $\mathbb{G}$ and there are no classical arbitrages, before and after τ

Indeed, under the positive density hypothesis, it can be proved that the probability $\mathbb{P}^*$, defined on $\mathbb{F} \vee \sigma(\tau)$ as

$$d\mathbb{P}^*|_{\mathcal{F}_t \vee \sigma(\tau)} = \frac{1}{\alpha_t(\tau)} d\mathbb{P}|_{\mathcal{F}_t \vee \sigma(\tau)}$$

satisfies the following assertions

(i) Under $\mathbb{P}^*$, τ is independent from $\mathcal{F}_t$ for any t

(ii) $\mathbb{P}^*|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$

(iii) $\mathbb{P}^*|_{\sigma(\tau)} = \mathbb{P}^*|_{\sigma(\tau)}$

It is now obvious that NFLVR hold in the enlarged filtration ($\mathbb{P}^*$ being a $\mathbb{G}$-e.m.m.).

Immersion setting

We recall that the filtration $\mathbb{F}$ is immersed in $\mathbb{G}$ if any $\mathbb{F}$ martingale is a $\mathbb{G}$ martingale. This is equivalent to

$$\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty)$$

Under the immersion assumption, all the three concepts of NA, NFLVR and NUPBR hold.

Let S be an $\mathbb{F}$ local martingale, then it is a $\mathbb{G}$ local martingale as well.

Emery's Example

Let S be defined through $dS_t = \sigma S_t dW_t$, where W is a Brownian motion.

Let $\tau = \sup\{t \leq 1 \, : \, S_1 - 2S_t = 0\}$, that is the last time before 1 when the price is equal to half of its terminal value at time 1.

In the above model NFLVR holds before τ. There are arbitrages after τ.

Note that

$$\{\tau \le t\} = \{\inf_{t \le s \le 1} 2\frac{S_s}{S_t} \ge \frac{S_1}{S_t}\}$$

therefore

$$\mathbb{P}(\tau \le t | \mathcal{F}_t) = \mathbb{P}(\inf_{t \le s \le 1} 2S_{s-t} \ge S_{1-t}) = \Phi(1-t)$$

where $\Phi(u) = \mathbb{P}(\inf_{s \le u} 2S_s \ge S_u)$. It follows that the Azéma supermartingale is a deterministic decreasing function, hence, τ **is a pseudo-stopping time,** hence S is a $\mathbb{G}$ martingale up to time τ and there are no arbitrages up to τ.

There are obviously arbitrages after τ, since, at time τ, one knows the value of S_1 and $S_1 > S_\tau$. In fact, for $t > \tau$, one has $S_t > S_\tau$, and the arbitrage occurs at any time before 1.

Honest times

A random time τ is **honest** if, for each $t \geq 0$, there exists an $\mathcal{F}_t$-measurable random variable τ_t such that $\tau = \tau_t$ on $\tau < t$.

In the case where $\tau = \sup\{t \le T,\ S_t = \sup_{s\le T} S_s\}$, one can find, in Dellacherie, Maisonneuve, Meyer (1992), *Probabilités et Potentiel, chapitres XVII-XXIV: Processus de Markov (fin), Compléments de calcul stochastique*, page 137 ***Par exemple, S_t peut représenter le cours d'une certaine action à l'instant t, et τ est le moment idéal pour vendre son paquet d'actions. Tous les spéculateurs cherchent à connaître τ sans jamais y parvenir, d'où son nom de variable aléatoire honnête.***

For instance, S_t may represent the price of some stock at time t and τ is the optimal time to liquidate a position in that stock. Every speculator strives to know when τ will occur, without ever achieving this goal. Hence, the name of honest random variable.

It is proved in Jeulin and in Jeulin and Yor that τ **is honest if and only if** $\widetilde{Z}_\tau = 1$. Moreover, $A^o_t = A^o_{t\wedge\tau}$.

Let τ be a finite honest time and assume that the market (S^0, S) is complete. Then, if τ is not an $\mathbb{F}$-stopping time, there are classical arbitrages before and after τ.

Before τ

From $m = \widetilde{Z} + A^o_-$ and $\widetilde{Z}_\tau = 1$, we deduce that $m_\tau \geq 1$.

Since τ is not a stopping time, $\mathbb{P}(A^o_{\tau-} > 0) > 0$.

The market being complete, the martingale m is the value of a self financing portfolio, with initial value 1, and $m_\tau = 1 + \int_0^\tau \varphi_s dS_s$ for a predictable φ. Since $m_t \geq 0$, the strategy φ is admissible.

After τ: Here, $t > \tau$

Using $m = \widetilde{Z} + A^o_-$, one obtains that $m_t - m_\tau = \widetilde{Z}_t - 1 + \Delta A^o_\tau$.

Consider the (finite) $\mathbb{G}$-stopping time

$$\nu := \inf\{t > \tau : \widetilde{Z}_t \leq \frac{1 - \Delta A^o_\tau}{2}\}.$$

Then,

$$m_\nu - m_\tau = \widetilde{Z}_\nu - 1 + \Delta A^o_\tau \leq \frac{\Delta A^o_\tau - 1}{2} \leq 0,$$

and, as τ is not an $\mathbb{F}$-stopping time,

$$\mathbb{P}(m_\nu - m_\tau < 0) = \mathbb{P}(\Delta A^o_\tau < 1) > 0.$$

Hence $-\int_\tau^{t\wedge\nu} \varphi_s dS_s = m_{\tau\wedge t} - m_{t\wedge\nu}$ is the value of a self-financing strategy with initial value 0 and terminal value $m_\tau - m_\nu \geq 0$ satisfying $\mathbb{P}(m_\tau - m_\nu > 0) > 0$.

From $m = Z + A^o$ and the fact that $A^o_t = A^o_{t\wedge\tau}$, one obtains that $m_t - m_\tau = Z_t - Z_\tau \geq -2$, hence the strategy is admissible.

Examples in a Brownian filtration

In this section, we assume that

$$S_t = \exp(\sigma W_t - \frac{1}{2}\sigma^2 t), \qquad \sigma > 0 \text{ given.}$$

- Consider the following random time (honest)

$$g := \sup\{t \,:\, S_t = a\},$$

where $0 < a < 1$. This time is well defined, since S_t goes to 0 when t goes to infinity.

Then $Z_t = 1 - (1 - \frac{S_t}{a})^+$, and

$$dZ_t = \mathbb{1}_{\{S_t < a\}} \frac{1}{a} dS_t - \frac{1}{2a} d\ell_t^a$$

Therefore,

$$\varphi := \frac{1}{a} \mathbb{1}_{\{S < a\}}$$

- Let, $S_t^* = \sup\{S_s, s \leq t\}$ and

$$\tau = \sup\{t \,:\, S_t = S_\infty^*\} = \sup\{t \,:\, S_t = S_t^*\}$$

Then, $Z_t = \frac{S_t}{S_t^*}$ and $dm_t = \frac{1}{S_t^*} dS_t$, therefore $\varphi_t = \frac{1}{S_t^*}$.

Example in a Poissonnian filtration

Let $dS_t = S_{t-}\psi dM_t$, $S_0 = 1$ with $\psi > 0$, where M is the compensated martingale of a Poisson process and τ given by

$$\tau := \sup\{t : \; S_t \geq b\} = \sup\{t : Y_t \leq a\}.$$

where $Y_t := \dfrac{\lambda\psi}{\ln(1+\psi)}t - N_t$, and $0 < b < 1$. Note that $S_t = e^{-\ln(1+\psi)Y_t}$. Then, the process

$$\varphi := \frac{\Psi(Y_- - a - 1)\mathbb{1}_{\{Y_- \geq a+1\}} - \Psi(Y_- - a)\mathbb{1}_{\{Y_- \geq a\}} + \mathbb{1}_{\{Y_- < a+1\}} - \mathbb{1}_{\{Y_- < a\}}}{\psi S_-},$$

where

$$\Psi(x) = \mathbb{P}(T^x < \infty), \qquad \text{with} \quad T^x = \inf\{t : x + Y_t < 0\}$$

On the one hand

$$Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t) = \Psi(Y_t - a) 1\!\!1_{\{Y_t \geq a\}} + 1\!\!1_{\{Y_t < a\}} = 1 + 1\!\!1_{\{Y_t \geq a\}} \left(\Psi(Y_t - a) - 1 \right).$$

On the other hand, setting $\theta = \frac{\mu}{\lambda} - 1$, one shows that the dual optional projection A^o of the process $1\!\!1_{[\tau,\infty)}$ equals

$$A^o = \frac{\theta}{1 + \theta} \sum_n 1\!\!1_{[\vartheta_n,\infty)},$$

where ϑ_n is the sequence of $\mathbb{F}$-stopping times defined by $\vartheta_1 = \inf\{t > 0 : Y_t = a\}$ and $\vartheta_n = \inf\{t > \vartheta_{n-1} : Y_t = a\}$.

Let $(A_t,\, t \geq 0)$ be an integrable increasing process (not necessarily $\mathbb{F}$-adapted). There exists a unique integrable $\mathbb{F}$-optional increasing process $(A^o_t, t \geq 0)$, called the dual optional projection of A such that

$$\mathbb{E}\left(\int_{[0,\infty[} Y_s dA_s\right) = \mathbb{E}\left(\int_{[0,\infty[} Y_s dA^o_s\right)$$

for any positive $\mathbb{F}$-optional process Y.

For any optional increasing process

$$\mathbb{E}(K_\tau) = \mathbb{E}(\sum \mathbb{1}_{\tau=\vartheta_n} K_{\vartheta_n}) = \mathbb{E}(\sum \mathbb{E}(\mathbb{1}_{\tau=\vartheta_n} | \mathcal{F}_{\vartheta_n}) K_{\vartheta_n})$$

and $\mathbb{E}(\mathbb{1}_{\tau=\vartheta_n} | \mathcal{F}_{\vartheta_n}) = \mathbb{P}(T^0 = \infty) = 1 - \Psi(0) = 1 - \frac{1}{1+\theta}$.

Random times constructed with hitting times

Brownian filtration

Suppose that $\mathbb{F}$ is the filtration generated by a Brownian motion W and, for $x > 0$,

$$T_x = \inf\{t, W_t \geq x\}.$$

Let $b > a > 0$, and consider the random time

$$\tau = \frac{1}{2}(T_a + T_b).$$

Then τ avoids $\mathbb{F}$ stopping times, and there are no classical arbitrages before τ.

The Azéma supermartingale associated with τ is

$$Z_t = \mathbb{1}_{\{T_a > t\}} + \mathbb{1}_{\{T_a \leq t\}} \Phi(t - T_a, b - W_t)$$

where $\Phi(s, x) = \frac{2}{\sqrt{2\pi s}} \int_0^x e^{-\frac{y^2}{2s}} dy$.

Then, denoting $\Phi'(s,x) = \frac{\partial}{\partial x}\Phi(s,x)$

$$m_t = 1 - \int_0^t 1\!\!1_{\{T_a \le s\}} \Phi'(s - T_a, b - W_s) dW_s$$

It follows that, on $t \le \tau$

$$W_t = \widehat{W}_t - \int_0^t 1\!\!1_{\{T_a < s\}} \frac{\Phi'}{\Phi - 1}(s - T_a, b - W_s) ds$$

Since $\mathbb{E}\left(\int_0^\tau 1\!\!1_{\{T_a<s\}} \left(\frac{\Phi'}{\Phi-1}(s-T_a, b-W_s)\right)^2 ds\right) < \infty$, there exists an e.m.m. and NFLVR holds.

It is not difficult to prove that $(\mathcal{H}')$ hypothesis holds for that example, even if τ is neither honest, has no density and immersion is not satisfied.

Poissonnian Filtration

Consider the random time $\tau = \frac{1}{2}(T_1 + T_2)$ that avoids $\mathbb{F}$-stopping times. Then the following properties hold:

(a) τ is not an honest time.

(b) $\widetilde{Z}_\tau = Z_\tau = e^{-\lambda \frac{1}{2}(T_2 - T_1)} < 1$,

(c) One can check that $m_\tau > 1$, hence there is a classical arbitrage before τ, given by

$$\varphi_t := -e^{-\lambda(t-T_1)} \left(1\!\!1_{\{N_{t-} \geq 1\}} - 1\!\!1_{\{N_{t-} \geq 2\}} \right) \frac{1}{\psi S_{t-}}.$$

NUPBR

A non-negative $\mathcal{K}_\infty$-measurable random variable ξ with $\mathbb{P}(\xi > 0) > 0$ yields an Unbounded Profit with Bounded Risk if for all $x > 0$ there exists an element $\theta^x \in \mathcal{A}_x^{\mathbb{K}}$ such that $V(x, \theta^x)_\infty := x + (\theta^x \centerdot S)_\infty \geq \xi$ $\mathbb{P}$-a.s. If there exists no such random variable we say that the financial market $\mathcal{M}(\mathbb{K})$ satisfies the No Unbounded Profit with Bounded Risk **(NUPBR)** condition.

A strictly positive $\mathbb{K}$-local martingale $L = (L_t)_{t\geq 0}$ with $L_0 = 1$ and $L_\infty > 0$ $\mathbb{P}$-a.s. is said to be a **local martingale deflator** in $\mathbb{K}$ on the time horizon $[0, \varrho]$ if the process LS^ϱ is an $\mathbb{K}$-local martingale; here ϱ is a $\mathbb{K}$-stopping time. If there exists a deflator, then NUPBR holds.

We recall that NFLVR=NA+NUPBR

NUPBR before τ

To any $\mathbb{F}$ local martingale X, we associate the $\mathbb{G}$ local martingale $\widehat{X}$ (stopped at time τ) defined as

$$\widehat{X}_t := X_t^\tau - \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s$$

Case of Continuous Filtration

If all $\mathbb{F}$ martingales are continuous, NUPBR holds before τ.

Let $\widehat{m}$ be the $\mathbb{G}$-martingale stopped at time τ associated with m, on $t \leq \tau$

$$\widehat{m}_t := m_t^\tau - \int_0^t \frac{d\langle m, m\rangle_s^{\mathbb{F}}}{Z_s}$$

and define a positive $\mathbb{G}$ local martingale L as $dL_t = -\frac{L_t}{Z_t} d\widehat{m}_t$. Recall that

$$\widehat{S}_t := S_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle S, m\rangle_s^{\mathbb{F}}}{Z_s}$$

is a $\mathbb{G}$ local martingale. From integration by parts, we obtain

$$\begin{aligned} d(LS^\tau)_t &= L_t dS_t^\tau + S_t dL_t + d\langle L, S^\tau\rangle_t^{\mathbb{G}} \\ &\overset{\mathbb{G}-\text{mart}}{=} L_t \frac{1}{Z_t} d\langle S, m\rangle_t^{\mathbb{F}} + \frac{1}{Z_{t-}} L_{t-} d\langle S, \widehat{m}\rangle_t^{\mathbb{G}} \\ &\overset{\mathbb{G}-\text{mart}}{=} L_t \frac{1}{Z_t} \left(d\langle S, m\rangle_t - d\langle S, m\rangle_t\right) = 0 \end{aligned}$$

Since SL is a $\mathbb{G}$-local martingale, NUPBR holds.

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Case of a Poisson Filtration

We assume that S is an $\mathbb{F}$ martingale of the form $dS_t = S_{t-}\psi_t dM_t$, with ψ is a predictable process, satisfying $\psi > -1$.

Let $Z_t = m_t - A^0_t$ be the optional decomposition of Z and $\widehat{m}$ the $\mathbb{G}$-martingale part of the $\mathbb{G}$ semi-martingale m. In a Poisson setting, from PRP, $dm_t = \nu_t dM_t$ for some predictable process ν, so that, on $t \leq \tau$,

$$d\widehat{m}_t = dm_t - \frac{1}{Z_{t-}} d\langle m \rangle_t = dm_t - \frac{1}{Z_{t-}} \lambda \nu_t^2 dt$$

In a Poisson setting, NUPBR holds before τ.

Indeed,

$$L = \mathcal{E}\left(-\frac{1}{Z_- + \nu} \centerdot \widehat{m}\right) = \mathcal{E}\left(-\frac{\nu}{Z_- + \nu} \centerdot \widehat{M}\right)$$

is a $\mathbb{G}$-local martingale deflator for S^τ

We are looking for a RN density of the form $dL_t = L_{t-}\kappa_t d\widehat{m}_t$ (and $\psi_t\kappa_t > -1$) so that L is positive and $S^\tau L$ is a $\mathbb{G}$ local martingale. Integration by parts formula leads to (on $t \le \tau$)

$$\begin{aligned} d(LS)_t &= L_{t-}dS_t + S_{t-}dL_t + d[L,S]_t \\ &\overset{\mathbb{G}-\text{mart}}{=} L_{t-}S_{t-}\psi_t \frac{1}{Z_{t-}} d\langle M, m\rangle_t + L_{t-}S_{t-}\kappa_t\psi_t\nu_t dN_t \\ &\overset{\mathbb{G}-\text{mart}}{=} L_{t-}S_{t-}\psi_t \frac{1}{Z_{t-}}\nu_t\lambda dt + L_{t-}S_{t-}\kappa_t\psi_t\nu_t\lambda(1+\frac{1}{Z_{t-}}\nu_t)dt \\ &= L_{t-}S_{t-}\psi_t\nu_t\lambda\left(\frac{1}{Z_{t-}} + \kappa_t(1+\frac{1}{Z_{t-}}\nu_t)\right)dt. \end{aligned}$$

Therefore, for $\kappa_t = -\frac{1}{Z_{t-}+\nu_t}$, one obtains a deflator. Note that

$$dL_t = L_{t-}\kappa_t d\widehat{m}_t = -L_{t-}\frac{1}{Z_{t-}+\nu_t}\nu_t d\widehat{M}_t$$

is indeed a positive martingale, since $\frac{1}{Z_{t-}+\nu_t}\nu_t < 1$.

Lévy processes

Assume that $S = \psi \star (\mu - \nu)$ where μ is the jump measure of a Lévy process and ν its compensator. Here, $\psi \star (\mu - \nu)$ is the process $\int_0^\cdot \int \psi(x,s)(\mu(dx,ds) - \nu(dx,ds))$. The martingale m admits a representation as $m = \psi^m \star (\mu - \nu)$. Then, the $\mathbb{G}$ compensator of μ is $\nu^{\mathbb{G}}$ where

$$\nu^{\mathbb{G}}(dt,dx) = \frac{1}{Z_{t-}} \left(Z_{t-} + \psi^m(t,x)\right) \nu(dt,dx)$$

i.e., S admits a $\mathbb{G}$-semi-martingale decomposition of the form

$$S = \psi \star (\mu - \nu^{\mathbb{G}}) - \psi \star (\nu - \nu^{\mathbb{G}})$$

Our goal is to find a positive martingale L of the form

$$dL_t = L_{t-}\kappa_t d\widehat{m}_t$$

so that LS is a local martingale.

From integration by parts formula

$$\begin{aligned} d(SL) &\overset{\mathbb{G}-\text{mart}}{=} -L_-\psi \star (\nu - \nu^{\mathbb{G}}) + d[S,L] = -L_-\psi \star (\nu - \nu^{\mathbb{G}}) + L_-\psi\psi^m\kappa \star \mu \\ &\overset{\mathbb{G}-\text{mart}}{=} -L_-\psi \star (\nu - \nu^{\mathbb{G}}) + L_-\psi\psi^m\kappa \star \nu^{\mathbb{G}} \\ &= -L_-\psi\left(1 - (1+\psi^m\kappa)\frac{1}{Z_-}(Z_- + \psi^m)\right) \star \nu \end{aligned}$$

Hence the possible choice $\kappa = -\frac{1}{Z_- + \psi^m}$. It can be checked that indeed, L is a positive martingale.

The positive $\mathbb{G}$-local martingale

$$L := \mathcal{E}\left(-\frac{\psi^m}{Z_- + \psi^m} I_{]\!]0,\tau]\!]} \star (\nu - \nu^{\mathbb{G}})\right)$$

$\mathbb{G}$-local martingale deflator for S^τ, and hence S^τ satisfies NUPBR.

General case, before τ

Let τ be a random time. Then, the following assertions are equivalent:

(i) The thin set $\{\widetilde{Z} = 0 \cap Z_{-} > 0\}$ is evanescent.

(ii) For any process S satisfying NUPBR($\mathbb{F}$), S^{τ} satisfies NUPBR($\mathbb{G}$).

After τ

We now assume that τ is a honest time, which satisfies $Z_\tau < 1$.

In Fontana et al. for a continuous filtration, it is proven that, if τ avoids $\mathbb{F}$ stopping times, arbitrages of the first kind exist after τ. The condition τ avoids $\mathbb{F}$ stopping times is equivalent to $Z_\tau = 1$

Case of Continuous Filtration

We start with the particular case of continuous martingales and prove that, for any honest time τ, NUPBR holds after τ.

Assume that τ is a honest time, which satisfies $Z_\tau < 1$ and that all $\mathbb{F}$ martingales are continuous. Then, for any honest time τ, NUPBR holds after τ. A deflator is given by $dL_t = -\frac{L_t}{1-Z_t} d\widehat{m}_t$.

The proof is based on Itô's calculus and the fact that, for any $\mathbb{F}$ martingale X (in particular for m and S)

$$\widehat{X}_t := X_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle X, m\rangle_s^{\mathbb{F}}}{Z_s} + \int_{t\wedge\tau}^t \frac{d\langle X, m\rangle_s^{\mathbb{F}}}{1 - Z_s}$$

is a $\mathbb{G}$ local martingale. Looking for a deflator of the form $dL_t = L_t \kappa_t d\widehat{m}_t$, and using integration by parts formula, we obtain that, for $\kappa = -(1-Z)^{-1}$, the process $L(S - S^\tau)$ is a local martingale.

Case of Continuous Filtration

We start with the particular case of continuous martingales and prove that, for any honest time τ, NUPBR holds after τ.

Assume that τ is a honest time, which satisfies $Z_\tau < 1$ and that all $\mathbb{F}$ martingales are continuous. Then, for any honest time τ, NUPBR holds after τ. A deflator is given by $dL_t = -\frac{L_t}{1-Z_t} d\widehat{m}_t$.

The proof is based on Itô's calculus and the fact that, for any $\mathbb{F}$ martingale X (in particular for m and S)

$$\widehat{X}_t := X_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle X, m\rangle_s^{\mathbb{F}}}{Z_s} + \int_{t\wedge\tau}^t \frac{d\langle X, m\rangle_s^{\mathbb{F}}}{1 - Z_s}$$

is a $\mathbb{G}$ local martingale. Looking for a deflator of the form $dL_t = L_t \kappa_t d\widehat{m}_t$, and using integration by parts formula, we obtain that, for $\kappa = -(1-Z)^{-1}$, the process $L(S - S^\tau)$ is a local martingale.

Case of a Poisson Filtration

We assume that S is an $\mathbb{F}$ martingale of the form $dS_t = S_{t-}\psi_t dM_t$, with ψ is a predictable process, satisfying $\psi > -1$.

The decomposition formula reads, after τ as

$$\widehat{S}_t = S_t + \int_{t\vee\tau}^{t} \frac{1}{1-Z_{s-}} d\langle S, m\rangle_s = S_t + \lambda \int_{t\vee\tau}^{t} \frac{1}{1-Z_{s-}} \nu_s \psi_s S_{s-} ds$$

Let $\mathbb{F}$ be a Poisson filtration and τ be an honest time satisfying $Z_\tau < 1$. Then, NUPBR holds after τ.

We are looking for a RN density of the form $dL_t = L_{t-}\kappa_t d\widehat{m}_t$ (and $\psi_t\kappa_t > -1$) so that L is positive $\mathbb{G}$ local martingale and $(S - S^\tau)L$ is a $\mathbb{G}$ local martingale. Integration by parts formula leads to

$$\begin{aligned}
d(L(S-S^\tau))_t &= L_{t-}d(S-S^\tau)_t + (S_{t-} - S^\tau_{t-})dL_t + d[L, S-S^\tau]_t \\
&\overset{\mathbb{G}-\text{mart}}{=} -\lambda L_{t-}S_{t-}\nu_t\psi_t \frac{1}{1-Z_{t-}}1\!\!1_{\{t>\tau\}}dt + L_{t-}S_{t-}\kappa_t\psi_t\nu_t 1\!\!1_{\{t>\tau\}}dN_t \\
&\overset{\mathbb{G}-\text{mart}}{=} -\lambda L_{t-}S_{t-}\nu_t\psi_t \frac{1}{1-Z_{t-}}1\!\!1_{\{t>\tau\}}dt \\
&\qquad +\lambda L_{t-}S_{t-}\kappa_t\psi_t\nu_t 1\!\!1_{\{t>\tau\}}(1 - \frac{1}{1-Z_{t-}}\nu_t)dt \\
&= \lambda L_{t-}S_{t-}\psi_t\nu_t 1\!\!1_{\{t>\tau\}}\left(-\frac{1}{1-Z_{t-}} + \kappa_t(1 - \frac{1}{1-Z_{t-}}\nu_t)\right)dt.
\end{aligned}$$

Therefore, for $\kappa_t = \frac{1}{1-Z_{t-}-\nu_t}$, one obtains a deflator.

Note that

$$dL_t = L_{t-}\kappa_t d\widehat{m}_t = L_{t-}\frac{1}{1 - Z_{t-} - \nu_t}\nu_t \mathbb{1}_{\{t>\tau\}} d\widehat{M}_t$$

is indeed a positive martingale, since $\frac{1}{1-Z_{t-}-\nu_t}\nu_t \Delta N_t > -1$.

$$L = \mathcal{E}\left(\frac{1}{1 - Z_- - \nu} 1\!\!1_{]\tau,\infty[} \centerdot \widehat{m}\right) = \mathcal{E}\left(\frac{\nu}{1 - Z_- - \nu} 1\!\!1_{]\tau,\infty[} \centerdot \widehat{M}\right)$$

is a $\mathbb{G}$ deflator

Lévy Processes

Assume that $S = \psi \star (\mu - \nu)$ where μ is the jump measure of a Lévy process and ν its compensator.

Then, after τ, the $\mathbb{G}$ compensator of μ is $\nu^{\mathbb{G}}$ where

$$\nu^{\mathbb{G}}(dt, dx) = \left(1 + 1\!\!1_{\{t \leq \tau\}} \frac{1}{Z_{t-}} \psi^m(t, x) - 1\!\!1_{\{t > \tau\}} \frac{1}{1 - Z_{t-}} \psi^m(t, x)\right) \nu(dt, dx)$$

i.e., S admits a $\mathbb{G}$-semi-martingale decomposition of the form

$$S = \psi \star (\mu - \nu^{\mathbb{G}}) - \psi \star (\nu - \nu^{\mathbb{G}})$$

Assume that τ be an honest time satisfying $Z_\tau < 1$ in a Lévy framework. Then, $S - S^\tau$ satisfies NUPBR.

Our goal is to find a positive martingale L of the form

$$dL_t = L_{t-}\kappa_t 1\!\!1_{\{t>\tau\}} d\widehat{m}_t$$

so that $L(S - S^\tau)$ is a local martingale.

From integration by parts formula

$$\begin{aligned}
d(L(S-S^\tau)) &\overset{\mathbb{G}-\text{mart}}{=} -L_- d(S-S^\tau) + d[S,L] \\
&= -L_-\psi\frac{\psi^m}{1-Z_-}1\!\!1_{]\tau,\infty[}\star\nu + L_-\kappa\psi\psi^m 1\!\!1_{]\tau,\infty[}\star\mu \\
&\overset{\mathbb{G}-\text{mart}}{=} -L_-\psi\frac{\psi^m}{1-Z_-}1\!\!1_{]\tau,\infty[}\star\nu + L_-\kappa\psi\psi^m 1\!\!1_{]\tau,\infty[}\star\nu^{\mathbb{G}} \\
&= -L_-\psi\psi^m 1\!\!1_{]\tau,\infty[}\left(-\frac{1}{1-Z_-} + \kappa(1-\frac{\psi^m}{1-Z_-})\right)\star\nu
\end{aligned}$$

Hence the possible choice $\kappa = \frac{1}{1-Z_- - \psi^m}$.

Consider the positive $\mathbb{G}$-local martingale

$$L := \mathcal{E}\left(\frac{\psi^m}{1 - Z_- - \psi^m} I_{]\!]\tau,\infty[\![} \star (\nu - \nu^{\mathbb{G}})\right)$$

L is a $\mathbb{G}$-martingale density for $S - S^\tau$.

General case after τ

Let τ be an honest time satisfying $Z_\tau < 1$. Then, the following assertions are equivalent:

(i) The thin set $\{\widetilde{Z} = 1 \cap Z_- < 1\}$ is evanescent.

(ii) For any process S such that $S - S^\tau$ satisfies NUPBR($\mathbb{F}$), $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).

Optional Integral

We recall the definition of the optional integral that will be of paramount importance in the last part of this paper. Let $\mathbb{K}$ be one of the filtrations $\{\mathbb{F}, \mathbb{G}\}$. Let X be a $\mathbb{K}$-martingale and H a (bounded) $\mathbb{K}$-optional process.

The compensated stochastic integral $M = H \odot X$ is the unique $\mathbb{K}$-local martingale such that, for any $\mathbb{K}$-local martingale Y,

$$\mathbb{E}\left([M, Y]_\infty\right) = \mathbb{E}\left(\int_0^\infty H_s \, d[X, Y]_s\right).$$

The process $[M, Y] - H \centerdot [X, Y]$ is an $\mathbb{K}$-local martingale.

In other terms, the compensated stochastic integral of H with respect to X is the unique local martingale, M, such that

$$M^c = {}^{p,\mathbb{K}}H \centerdot X^c \quad \text{and} \quad \Delta M = H\Delta X - {}^{p,\mathbb{K}}(H\Delta X)$$

where ${}^{p,\mathbb{K}}U$ denotes the $\mathbb{K}$-predictable projection of the process U.

The Case of Quasi-Left Continuous Processes

NUPBR before τ

We assume that m is quasi continuous on left and that $\tilde{Z} > 0$.

We prove that, in this case, NUPBR is preserved under random horizon. Define the process

$$\widetilde{N} := -\frac{1}{\widetilde{Z}} \odot \widehat{m} = -\frac{1}{\widetilde{Z}} 1\!\!1_{]0,\tau]} \odot \left(m - \frac{1}{Z_-} 1\!\!1_{]0,\tau]} \centerdot \langle m \rangle^{\mathbb{F}} \right).$$

(a) The process $\mathcal{E}(\widetilde{N})$ is a positive $\mathbb{G}$-martingale.

(b) The process $\mathcal{E}\left(\widetilde{N}\right) S^\tau$ is a $\mathbb{G}$-local martingale.

NUPBR after τ

Assume that $Z_\tau < 1, 0 < \tilde{Z} < 1$ and the martingale m is quasi left continuous. We define the process

$$\widetilde{N} := 1\!\!1_{]\tau,\infty[}\frac{1}{1-\widetilde{Z}} \odot \widehat{m} = \frac{1}{1-\widetilde{Z}} 1\!\!1_{]\tau,\infty[} \odot \left(m - \frac{1}{1-Z_-} 1\!\!1_{]\tau,\infty[} \centerdot \langle m \rangle^{\mathbb{F}} \right).$$

Then,

(a) The process $\mathcal{E}(\widetilde{N})$ is a positive $\mathbb{G}$-martingale.

(b) The process $\mathcal{E}\left(\widetilde{N}\right)(S - S^\tau)$ is a $\mathbb{G}$-local martingale.

A (finite) random time τ is a strict honest time (i.e., $[\![\tau]\!] \cap [\![T]\!] = \emptyset$ for any $\mathbb{F}$-stopping time T) if and only if $Z_\tau = 1$ a.s. on $(\tau < \infty)$.

Assume that τ is a strict honest time. From $\tilde{Z}_\tau = 1$ and using the continuity of A^o, the relation $\widetilde{Z} = m - A^o_-$ leads to the result.

Assume now that $Z_\tau = 1$. We have $1 = Z_\tau \leq \widetilde{Z}_\tau \leq 1$, so $\widetilde{Z}_\tau = 1$ and τ is an honest time. Furthermore, as $\Delta A^o_\tau = \widetilde{Z}^\tau_\tau - Z^\tau_\tau = 0$, for each $\mathbb{F}$ stopping time T we have

$$\mathbb{P}(\tau = T < \infty) = \mathbb{E}(1\!\!1_{\{\tau = T\}} 1\!\!1_{\{\Delta A^o_\tau = 0\}} 1\!\!1_{(T<\infty)}) = \mathbb{E}(\int_0^\infty 1\!\!1_{\{u=T\}} 1\!\!1_{\{\Delta A^o_u = 0\}} dA^o_u) = 0.$$

So τ is a strict honest time.

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My goal is not to know the answers, I am trying to understand the questions.

Confucius

Thank you for your attention