On Bid-Ask Prices For Dividend Paying Securities

Pricing and Hedging via Dynamic Coherent Acceptability Indices

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- Introduction and overview of the existing methods

- Main Ingredients:
  Dynamic Coherent Acceptability Indices
  Dynamic Coherent Risk Measures

- No-Good-Deals and Fundamental Theorem of Assets Pricing

- Good-deal ask and bid prices

- Examples

Introduction

Incomplete Financial Market Models

- not every contingent claim (financial contract) is hedgeable
- there are infinitely many risk-neutral probability measures (assuming no arbitrage)
- fair price $= \mathbb{E}^{\mathbb{Q}}[\text{Payoff}^*]$, where $\mathbb{Q}$ is a risk-neutral probability
- One can compute **no arbitrage bounds** for prices

$$\Big[\inf_{\mathbb{Q}\in\mathcal{R}} \mathbb{E}^{\mathbb{Q}}[\text{Payoff}^*], \sup_{\mathbb{Q}\in\mathcal{R}} \mathbb{E}^{\mathbb{Q}}[\text{Payoff}^*]\Big]$$

where $\mathcal{R}$ is the set of all risk neutral probabilities

Introduction

- **No arbitrage bounds** for prices

$$\Big[\inf_{\mathbb{Q}\in\mathcal{R}} \mathbb{E}^{\mathbb{Q}}[\text{Payoff}^*], \sup_{\mathbb{Q}\in\mathcal{R}} \mathbb{E}^{\mathbb{Q}}[\text{Payoff}^*]\Big]$$

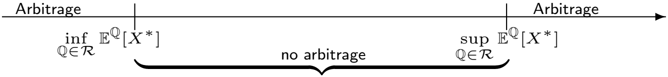
  where $\mathcal{R}$ is the set of all risk neutral probabilities

- The **no arbitrage bounds are too wide** in practice
- Shrinking the arbitrage free price interval:

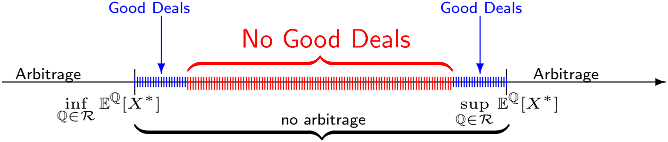
  (a) Indifference pricing via utility maximization - a price at which an agent receives the same expected utility between trading and not trading (book by Carmona '09). Limitations: numerical implementations and explicit calculations may not be robust; resulting bid and ask prices are not necessarily risk-neutral in practice (Staum '07)

  (b) Rule out deals that are **too good to be true**, eliminating prices with high Sharpe ratios (Cochrane and Requejo '99)

Methodology



Methodology



- Similar to arbitrage opportunities, **good-deals** should not be available in the market, since everyone would be willing to take them
- A market maker wants to determine the best bid and ask prices to offer to the market for a specified **acceptability level** that he/she picks
- Easy to compute, robust, does not contradict the general arbitrage theory

Literature review

- Bernardo and Ledoit '00 and Pinar, Salih, and Camci '10 - cash flows are good deals if the Gain-Loss Ratio is high
- Good-deal-bound approach has been generalized and used in applications by Carr et al. '01, Jaschke and Kuchler '01, Staum '04, Engwerda et al. '05, Bjork and Slinko '06, Kloppel and Schweitzer '07, Arai and Fukasawa '11.
- Using coherent risk measures Cherny and Madan '06, '07
- Dynamic bid and ask prices via dynamic risk measures
- **Conic Finance** - **static** bid and ask prices by **Acceptability Indices** - Madan and Cherny '11;
  no bid/ask allowed for hedgeable securities

Our contribution

- Extend Good-Deal bound setup using

  Dynamic (Coherent) Acceptability Indices (DCAI)

  and

  Dynamic (Coherent) Risk Measures (DCRM)

- Price (bid/ask) contingent claims that pay dividends (CDS, IRS); the input is a process

- Allow transaction cost for underlying/headgeable securities

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Dynamic Conic Finance
Keyword - **DYNAMIC**

Notations, General Assumptions

- T-fixed time horizon, $\mathcal{T} := \{0, 1, \dots, T\}$.
- $(\Omega, \mathcal{F}_T, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$ - the underlying filtered probability space. Assume Ω finite for this talk.
- $L^0 := L^0(\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P})$ - the set of all $\mathbb{F}$-adapted processes.
- $\{B_t\}_{t\in\mathcal{T}}$ - banking account.
- Any $D \in L^0$ is interpreted as a *dividend stream*.
- $D^* := \frac{D}{B}$ - *discounted dividend stream*.

Part I:

Dynamic Coherent Acceptability Indices (DCAI)

and

Dynamic Coherent Risk Measures (DCRM)

Dynamic Coherent Acceptability Index (DCAI):

$\alpha : \mathcal{T} \times L^0 \times \Omega \to [0, +\infty]$ that satisfies the following properties:

1. **Monotonicity.** If $D_s \geq D'_s$ for all $s \geq t$, then $\alpha_t(D) \geq \alpha_t(D')$
2. **Scale invariance.** $\alpha_t(\lambda D) = \alpha_t(D)$ for all $\lambda > 0$
3. **Quasi-concavity**. If $\alpha_t(D, \omega) \geq x$ and $\alpha_t(D', \omega) \geq x$, then $\alpha_t(\lambda D + (1-\lambda)D', \omega) \geq x$ for all $\lambda \in [0, 1]$
4. **Adaptiveness.** $\alpha_t(D)$ is $\mathcal{F}_t$-measurable
5. **Independence of the past.** If $1_A D_s = 1_A D'_s$ for $A \in \mathcal{F}_t$ and for all $s \geq t$, then $1_A \alpha_t(D) = 1_A \alpha_t(D')$
6. **Translation invariance.** $\alpha_t(D + m1_{\{t\}}) = \alpha_t(D + m1_{\{s\}} \frac{B_s}{B_t})$, $s \geq t$
7. **Dynamic consistency.** Let $D \in \mathcal{D}$, and $m \geq 0$ be $\mathcal{F}_t$ measurable
   (a) If $D_t \geq 0$ and $\alpha_{t+1}(D) \geq m$, then $\alpha_t(D) \geq m$
   (b) If $D_t \leq 0$ and $\alpha_{t+1}(D) \leq m$, then $\alpha_t(D) \leq m$

Dynamic Coherent Acceptability Index (DCAI): Discussion

- **Monotone, Quasi-concave, Unitless**
- Generalization of Sharpe Ratio $\mathrm{SR}(X) = \mathbb{E}[X - r]/\mathrm{Std}[X]$
  SR is not monotone
- Main Example:
  Dynamic Gain-Loss Ratio

$$\mathrm{dGLR}_t(D)(\omega) := \begin{cases} \dfrac{\mathbb{E}_t^{\mathbb{P}}[\sum_{s=t}^T D_s^*](\omega)}{\mathbb{E}_t^{\mathbb{P}}\Big[\Big(\sum_{s=t}^T D_s^*\Big)^-\Big](\omega)}, & \text{if } \mathbb{E}_t^{\mathbb{P}}\Big[\displaystyle\sum_{s=t}^T D_s^*\Big](\omega) > 0 \\ 0, & \text{otherwise} \end{cases}$$

for all $t \in \mathcal{T}$, and $\omega \in \Omega$.
Static $\mathrm{GLR}(X) = \mathbb{E}[X]/\mathbb{E}[X^-]$

Dynamic Coherent Risk Measure (DCRM):

$\rho : \mathcal{T} \times L^0 \times \Omega \to \mathbb{R}$ that satisfies the following properties:

1. **Monotonicity.** If $D_s \geq D'_s$ for all $s \geq t$, then $\rho_t(D) \leq \rho_t(D')$
2. **Homogeneity.** $\rho_t(\lambda D) = \lambda \rho_t(D)$ for all $\lambda > 0$
3. **Subadditivity.** $\rho_t(D + D') \leq \rho_t(D) + \rho_t(D')$
4. **Adaptiveness.** $\rho_t(D)$ is $\mathcal{F}_t$-measurable
5. **Independence of the past.** If $1_A D_s = 1_A D'_s$ for $A \in \mathcal{F}_t$ and for all $s \geq t$, then $1_A \rho_t(D) = 1_A \rho_t(D')$
6. **Cash Additivity.** $\rho_t(D + m 1_{\{s\}} \frac{B_s}{B_t}) = \rho_t(D) - m$, $s \geq t$
7. **Dynamic consistency.** For all $A \in \mathcal{F}_t, D \in L^0$

$$1_A (\min_{\omega \in A} \rho_{t+1}(D, \omega) - D_t) \leq 1_A \rho_t(D) \leq 1_A (\max_{\omega \in A} \rho_{t+1}(D, \omega) - D_t)$$

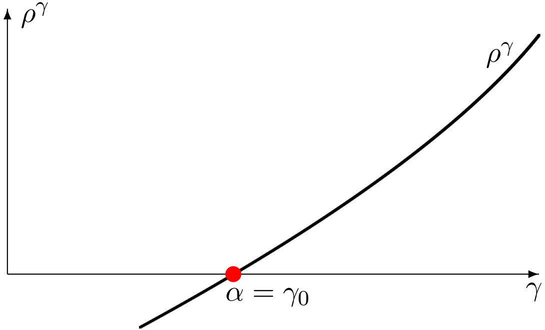
Generalization of Value At Risk V@R; Measured in monetary units ($)

Duality and Robust Representations for DCAIs

$$\alpha \quad \longleftrightarrow \quad \{\rho^\gamma\}_{\gamma\in\mathbb{R}_+}$$

$$\text{DAI} \quad \longleftrightarrow \quad \text{increasing family of DCRMs}$$

$$\alpha_t(D)(\omega) = \sup\{\gamma \in (0,\infty) : \rho_t^\gamma(D)(\omega) \le 0\}$$



Theorem (Bielecki, C., Zhang, 2012)

- *If α is a normalized, right-continuous DCAI then there exists a left-continuous and increasing family of DCRMs such that*

$$\alpha_t(D)(\omega) = \sup\{\gamma \in (0,\infty) : \rho_t^\gamma(D)(\omega) \leq 0\},$$

for all $\omega \in \Omega,\ t \in \mathcal{T},\ D \in L^0$.

- *If $(\rho^\gamma)_{\gamma\in(0,\infty)}$ is a left-continuous and increasing family of DCRMs, then there exists a right-continuous and normalized DCAI α such that*

$$\rho_t^\gamma(D)(\omega) = \inf\{c \in \mathbb{R} : \alpha_t(D + \delta_t^*(1c))(\omega) \geq \gamma\},$$

for all $\omega \in \Omega,\ t \in \mathcal{T},\ D \in L^0$.

Duality for Risk Measures

For a fixed $\gamma \in \mathbb{R}_+$

$$\text{risk measure } \rho^\gamma \quad \longleftrightarrow \quad \{\mathcal{Q}_t^\gamma\}_{t=0}^T \text{ a set of probability measures}$$

$\mathcal{Q}^\gamma = \{\mathcal{Q}_t^\gamma\}_{t=0}^T$ increasing, dynamic consistent sets of probabilities

Theorem (Robust Representation of DCRMs; BCZ'12)

A function ρ^γ is a DCRM if and only if there exists $(\mathcal{Q}_t^\gamma)_{t=0}^T$ such that,

$$\rho_t^\gamma(D) = -\inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{s=t}^T D_s^*\Big]$$

Theorem (Robust Representation of DCAIs; BCZ'12)

α is a DCAI if and only if there exists $((\mathcal{Q}_t^\gamma)_{t=0}^T)_{\gamma\in(0,\infty)}$ such that

$$\alpha_t(D)(\omega) = \sup\Big\{\gamma \in (0,\infty) : \inf_{\mathbb{Q}\in\mathcal{Q}_t^\gamma} \mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{s\geq t} D_s^*\Big](\omega) \geq 0\Big\}$$

Dynamically Consistent Sets of Probability Measures

Definition

(i) A sequence of sets of probability measures $(\mathcal{Q}_t)_{t=0}^T$ absolutely continuous with respect to $\mathbb{P}$ is called **dynamically consistent with respect to the filtration** $(\mathcal{F}_t)_{t=0}^T$ if the sequence is of full-support and the following inequality holds true

$$\mathbb{1}_E \min_{\omega\in E}\left\{\inf_{\mathbb{Q}\in\mathcal{Q}_{t+1}} \mathbb{E}_{t+1}^{\mathbb{Q}}[X](\omega)\right\} \leq \mathbb{1}_E \inf_{\mathbb{Q}\in\mathcal{Q}_t} \mathbb{E}_t^{\mathbb{Q}}[X] \leq \mathbb{1}_E \max_{\omega\in E}\left\{\inf_{\mathbb{Q}\in\mathcal{Q}_{t+1}} \mathbb{E}_{t+1}^{\mathbb{Q}}[X](\omega)\right\}$$

for all $t \in \{0, 1, \ldots, T-1\}$, $E \in \mathcal{F}_t$, and $\mathcal{F}_T$-measurable random variables X.

(ii) A family of sequences of sets of probability measures $((\mathcal{Q}_t^\gamma)_{t=0}^T)_{\gamma\in(0,\infty)}$ is called **increasing** if $\mathcal{Q}_t^\gamma \supseteq \mathcal{Q}_t^\beta$, for all $\gamma \geq \beta > 0$ and $t \in \mathcal{T}$.

Part I: Bottom Line

$$\alpha \quad \longleftrightarrow \quad \{\rho^\gamma\}_{\gamma\in\mathbb{R}_+} \quad \longleftrightarrow \quad \{\mathcal{Q}_t^\gamma\}_{t=0,\gamma\in\mathbb{R}_+}^T$$

General Remarks on Performance and Risk Measures

- Coherent Risk Measures can be replaced by Convex Risk Measures
  - Homogeneity, Subadditivity $\twoheadrightarrow$ Convexity;
  - Characterize such risk measures using conditional g-Expectations and Backward Stochastic Difference Equations (with convex drivers) on general probability space;
  - Corresponding DAI will be sub-scale invariant $\alpha_t(\lambda D) \geq \alpha_t(D),\ \lambda \in (0,1]$;
  - Bielecki, C., Chen '13 and S. Biagini, J. Bion-Nadal '13

- Dynamic Assessment Indices - generalization of DAI and DRM;
  - Monotone, Quasi-Concave and Local. Defined on an L^0-module;
  - Bielecki, C., Drapeau and Karliczek '13. Complete dual characterization using theory of L^0-modules;
  - Could be path dependent;
  - Robust representation for strongly time consistent DAI using certainty equivalent;
  - See also M. Frittelli and M. Maggis '11.

- Dynamic Consistency, in general, is a delicate point.

Part II: Dynamic Conic Finance

Financial Market Model: underlying assets

- $B := \left(\left(\prod_{s=0}^{t} (1+r_s) \right) \right)_{t=0}^{T}$ is the savings account.
- $P^{ask} := \left((P_t^{ask,1}, \dots, P_t^{ask,N}) \right)_{t=0}^{T}$ is the ex-dividend price process ; $A^{ask} := \left((A_t^{ask,1}, \dots, A_t^{ask,N}) \right)_{t=1}^{T}$ is the associated (cumulative) dividend process.
- $P^{bid} := \left((P_t^{bid,1}, \dots, P_t^{bid,N}) \right)_{t=0}^{T}$ is the ex-dividend price process; $A^{bid} := \left((A_t^{bid,1}, \dots, A_t^{bid,N}) \right)_{t=1}^{T}$ is the cumulative dividend process.
- **Natural Assumption:** $P_t^{ask} \geq P_t^{bid}, \ \Delta A_t^{bid} \geq \Delta A_t^{ask}$.

Goals

- build up an arbitrage free theory for this market
- use DCAI, DCRM and good-deal bounds to produce consistent bid/ask prices for contingent claims in this market

Example: Credit Default Swap (CDS) contract

- τ be the random time of the credit event of the reference entity
- Initiation date $t = 0$, maturity $t = T$, nominal value of $1, and the loss-given-default $\delta \geq 0$ paid at default
- κ^{bid} is the CDS spread quoted by the dealer to sell protection
- κ^{ask} is the CDS spread quoted by the dealer to buy protection
- For the CDS contract specified above,

$$A_t^{ask} := 1_{\{\tau \leq t\}} \delta - \kappa^{ask} \sum_{u=1}^{t} 1_{\{u < \tau\}},$$

$$A_t^{bid} := 1_{\{\tau \leq t\}} \delta - \kappa^{bid} \sum_{u=1}^{t} 1_{\{u < \tau\}}, \qquad t \in \mathcal{T}$$

- P_t^{ask} and P_t^{bid} are the mark-to-market prices of the CDS

The Value Process and the Self-Financing Condition

Definition

The **value process** $V(\phi)$ associated with a trading strategy ϕ is defined as

$$V_t(\phi) = \begin{cases} \phi_1^0 + \sum_{j=1}^N \mathbb{1}_{\{\phi_1^j \geq 0\}} \phi_1^j P_0^{ask,j} + \sum_{j=1}^N \mathbb{1}_{\{\phi_1^j < 0\}} \phi_1^j P_0^{bid,j}, & \text{if } t = 0, \\ \phi_t^0 B_t + \sum_{j=1}^N \mathbb{1}_{\{\phi_t^j \geq 0\}} \phi_t^j (P_t^{bid,j} + \Delta A_t^{ask,j}) & \\ \quad + \sum_{j=1}^N \mathbb{1}_{\{\phi_t^j < 0\}} \phi_t^j (P_t^{ask,j} + \Delta A_t^{bid,j}), & \text{if } 1 \leq t \leq T. \end{cases}$$

A trading strategy ϕ is **self-financing** if for all $t = 1, 2, \ldots, T-1$

$$B_t \Delta\phi_{t+1}^0 + \sum_{j=1}^N P_t^{ask,j} \mathbb{1}_{\{\Delta\phi_{t+1}^j \geq 0\}} \Delta\phi_{t+1}^j + \sum_{j=1}^N P_t^{bid,j} \mathbb{1}_{\{\Delta\phi_{t+1}^j < 0\}} \Delta\phi_{t+1}^j$$
$$= \sum_{j=1}^N \phi_t^j \mathbb{1}_{\{\phi_t^j \geq 0\}} \Delta A_t^{ask,j} + \sum_{j=1}^N \phi_t^j \mathbb{1}_{\{\phi_t^j < 0\}} \Delta A_t^{bid,j}.$$

For Arbitrage, Risk Neutral Probabilities, First Fundamental Theorem of Asset Pricing, Consistent Price System - see Bielecki, C., Rodriguez, forthcoming in Math Fin.

Hedging cash flows at zero cost

- Set of self-financing trading strategies initiated at time t:

$$\mathcal{S}(t) := \begin{cases} \{\phi : \phi \ \text{ is s.f.}, V_0(\phi) = 0\}, & t = 0 \\ \{\phi : \phi \ \text{ is s.f.}, \ \phi_s = \mathbb{1}_{\{s \geq t+1\}}\phi_s, \ s = 1, 2, \ldots, T\}, & t = 1 \ldots, T-1 \end{cases}$$

Any $\phi \in \mathcal{S}(t)$ is of the form $(0, \ldots, 0, \phi_{t+1}, \phi_{t+2}, \ldots, \phi_T)$

- Set of hedging cash flows initiated at time t:

$$\mathcal{H}^0(t) := \Big\{ \Big(0, \ldots, 0, \Delta V^*_{t+1}(\phi), \ldots, \Delta V^*_T(\phi)\Big) \ : \ \phi \in \mathcal{S}(t) \Big\}$$

for $t \in \{0, \ldots, T-1\}$, where $V^*(\phi)$ is the discounted wealth process, and $\Delta V^*_{t+1} = V^*_{t+1} - V^*_t$.

Notations

$$\mathcal{L}_+(t) := \Big\{ (Z_s)_{s=0}^T : Z_s \in L_+(\Omega, \mathcal{F}_s, \mathbb{P}), \ Z_s = \mathbb{1}_{\{s \geq t+1\}} Z_s, s = 0, \ldots, T \Big\},$$

$$\mathcal{H}(t) := \Big\{ \Big(0, \ldots, 0, \Delta(V^*_{t+1}(\phi) - Z_{t+1}), \ldots, \Delta(V^*_T(\phi) - Z_T)\Big) \ : \ \phi \in \mathcal{S}(t), \ Z \in \mathcal{L}_+(t)$$

No-arbitrage condition and risk-neutral measures

Definition

An **arbitrage opportunity** at time $t \in \{0, \dots, T-1\}$ for $\mathcal{H}(t)$ is a cash flow $H \in \mathcal{H}(t)$ such that $\sum_{s=t}^{T} H_s(\omega) \geq 0$ for all $\omega \in \Omega$, and $\mathbb{E}_t^{\mathbb{P}}[\sum_{s=t}^{T} H_s](\omega) > 0$ for some $\omega \in \Omega$.

A probability measure $\mathbb{Q}$ is **risk-neutral for** $\mathcal{H}(t)$ if $\mathbb{Q} \sim \mathbb{P}$, and if $\mathbb{E}_t^{\mathbb{Q}}[\sum_{s=t}^{T} H_s](\omega) \leq 0$ for all $\omega \in \Omega$ and all $H \in \mathcal{H}(t)$.

- $\mathcal{R}(\mathcal{H}(t))$ is the set of all risk-neutral measures
- NA has the usual interpretation of "not making something out of nothing"
- NA and Risk-Neutral agree with classical theory

No-good-deal condition

Definition

The **No-Good-Deal (NGD)** condition holds true for $\mathcal{H}(t)$ at time $t \in \{0, \ldots, T-1\}$ and level $\gamma > 0$ if $\rho_t^\gamma(H)(\omega) \geq 0$ for all $H \in \mathcal{H}(t)$ and $\omega \in \Omega$

- $\{\rho^\gamma\}_{\gamma \in \mathbb{R}_+}$ - a family of DCRMs
- The no-good-deal condition for different times are related through the dynamical consistency property of ρ^γ
- If NGD is satisfied for $\gamma > 0$, then it is also satisfied for $\gamma' > \gamma$ since ρ^γ is increasing in γ
- NGD depends on the choice of ρ^γ

Fundamental theorem of NGD pricing

Theorem (BCIR, BCC)

NGD is satisfied for $\mathcal{H}(t)$ at time $t \in \{0, \ldots, T-1\}$ and level $\gamma > 0$ if and only if $\mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma \neq \emptyset$.

- If NGD is satisfied, then NA is also satisfied.
- Necessity is an immediate consequence of the robust representation theorem for DCRMs.
- Sufficiency is more involved. Requires a separation argument and the duality theorem between DCAIs and DCRMs.

Good-deal ask and bid prices

Let $\delta_t^+, \delta_t : L^0 \to L^0$ as follows

$$\begin{aligned} \delta_t^+(D) &:= \big(0, \ \ldots, \ 0, \ 0, \ D_{t+1}, \ \ldots \ D_T\big), \qquad t \in \{0, \ldots, T-1\} \\ \delta_t(D) &:= \big(0, \ \ldots, \ 0, \ D_t, \ 0, \ \ldots, \ 0\big), \qquad t \in \mathcal{T} \end{aligned}$$

Definition

The discounted *good-deal ask and bid prices* of a derivative contract D, at level $\gamma > 0$, at time $t \in \{1, \ldots, T-1\}$ are defined as

$$\Pi_t^{ask,\gamma}(D)(\omega) : = \inf\{v \in \mathbb{R} : \exists\ H \in \mathcal{H}(t) \text{ s.t. } \alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*))(\omega) \geq \gamma\}$$

$$\Pi_t^{bid,\gamma}(D)(\omega) : = \sup\{v \in \mathbb{R} : \exists\ H \in \mathcal{H}(t) \text{ s.t. } \alpha_t(\delta_t^+(D^*) + H - \delta_t(\mathbf{1}v))(\omega) \geq \gamma\}$$

- Prices at different times are related by the dynamic consistency property of α
- We have that $\Pi_t^{ask,\gamma}(D) = -\Pi_t^{bid,\gamma}(-D)$
- Prices depend on α, level γ, and hedging cash flows $\mathcal{H}(t)$

Theorem (BCIR, BCC)

The discounted good-deal ask and bid prices of a derivative contract $D \in L^0$, at level $\gamma > 0$, at time $t \in \{1, \ldots, T-1\}$ satisfy

$$\Pi_t^{ask,\gamma}(D) = \sup_{\mathbb{Q} \in \mathcal{Q}_t^\gamma \cap \mathcal{R}(\mathcal{H}(t))} \mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^*\Big]$$

$$\Pi_t^{bid,\gamma}(D) = \inf_{\mathbb{Q} \in \mathcal{Q}_t^\gamma \cap \mathcal{R}(\mathcal{H}(t))} \mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} D_s^*\Big]$$

- $\Pi_t^{ask,\gamma}(D)$ and $\Pi_t^{bid,\gamma}(D)$ are risk-neutral
- Since $\mathcal{Q}_t^\gamma$ is $\uparrow$ in γ, we have that $\Pi_t^{ask,\gamma}(D) - \Pi_t^{bid,\gamma}(D)$ is $\uparrow$ in γ
- $\Pi_t^{ask,\gamma}(D) \geq \Pi_t^{bid,\gamma}(D)$

Superhedging prices

Discounted superhedging ask and bid prices are given as

$$S_0^{bid}(D) = \inf_{\mathbb{Q}\in\mathcal{R}(\mathcal{H}(0))} \mathbb{E}^{\mathbb{Q}}[\sum_{s=1}^{Y} D_s^*]$$

$$S_0^{ask}(D) = \sup_{\mathbb{Q}\in\mathcal{R}(\mathcal{H}(0))} \mathbb{E}^{\mathbb{Q}}[\sum_{s=1}^{Y} D_s^*]$$

Shrinking superhedging price interval

$$S_0^{bid}(D) \leq \Pi_0^{bid,\gamma}(D) \leq \Pi_0^{ask,\gamma}(D) \leq S_0^{ask}(D)$$

Forward ask and bid prices

Assume that the risk-free rate is deterministic.

Definition

The *good-deal ask and bid forward prices*, with delivery at time T, written at time $t \in \{1, \dots, T-1\}$, of a derivative contract $D \in L^0$, at level $\gamma > 0$ are defined as

$$
\begin{aligned}
F_t^{ask,\gamma,T}(D)(\omega) := \inf\{ & f \in \mathbb{R} : \ \exists\, H \in \mathcal{H}(t) \text{ s.t.} \\
& \alpha_t(\delta_T(\mathbf{1} B_T^{-1} f) + H - \delta_t^+(D^*))(\omega) \geq \gamma\}, \\
F_t^{bid,\gamma,T}(D)(\omega) := \sup\{ & f \in \mathbb{R} : \ \exists\, H \in \mathcal{H}(t) \text{ s.t.} \\
& \alpha_t(-\delta_t(\mathbf{1} B_T^{-1} f) + H + \delta_t^+(D^*))(\omega) \geq \gamma\}
\end{aligned}
$$

for all $\omega \in \Omega$.

- The case in which r is random is much harder

Representation theorem for forward good-deal ask and bid prices

Theorem (BCIR)

The good-deal ask and bid forward prices of a derivative contract $D \in L^0$, with delivery at time T, written at time $t \in \{1, \dots, T-1\}$ and level $\gamma > 0$, satisfy

$$F_t^{ask,\gamma,T}(D)(\omega) = B_T \Pi_t^{ask,\gamma}(D),$$
$$F_t^{bid,\gamma,T}(D)(\omega) = B_T \Pi_t^{bid,\gamma}(D).$$

Definition

$$\mathrm{dGLR}_t(D)(\omega) := \begin{cases} \dfrac{\mathbb{E}_t^{\mathbb{P}}[\sum_{s=t}^T D_s^*](\omega)}{\mathbb{E}_t^{\mathbb{P}}\Big[\Big(\sum_{s=t}^T D_s^*\Big)^-\Big](\omega)}, & \text{if} \quad \mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=t}^T D_s^*\Big](\omega) > 0, \\ 0, & \text{otherwise}, \end{cases}$$

for all $t \in \mathcal{T}$, and $\omega \in \Omega$.

- It is shown in Bielecki, C., Zhang (forthcoming in math fin) that the dGLR is a dynamic coherent acceptability index.
- NA is satisfied at time $t \in \mathcal{T}$ if and only if $\mathrm{dGLR}_t(H)$ is bounded for all $H \in \mathcal{H}(t)$.

Correspondence

Define the family of sets of probability measures $\{\widehat{\mathcal{Q}}^{\gamma},\ \gamma > 0\}$ as

$$\widehat{\mathcal{Q}}^{\gamma} := \Big\{\mathbb{Q} : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = c(1+\Lambda),\ c > 0,\ \Lambda \in \mathfrak{L}^{\gamma},\ c\,\mathbb{E}^{\mathbb{P}}[1+\Lambda] = 1\Big\},$$

where

$$\mathfrak{L}^{\gamma} := \{\Lambda\ :\ \Lambda \text{ is an } \mathcal{F}_T\text{-measurable r.v.},\ 0 \leq \Lambda \leq \gamma\}$$

- $\widehat{\mathcal{Q}}$ does not depend on time
- $\widehat{\mathcal{Q}}$ is an increasing family of dynamically consistent sets of probability measures that corresponds/generates dGLR
- It satisfies all necessary technical assumptions for NGD theory above

Market Model Set-Up

Table: *Bid price paths of underlying security* P^{bid}

ω	$t=0$	$t=1$	$t=2$
ω_1	50	80	90
ω_2	50	80	70
ω_3	50	80	60
ω_4	50	40	60
ω_5	50	40	30

- Ask price of underlying security: $P^{ask} := (1+\lambda)P^{bid}$, where λ is the *transaction cost coefficient*.
- Consider Arithmetic Asian European style call option with strike 75:

$$\left(\frac{1}{3}\left(P_0^{mid} + P_1^{mid} + P_2^{mid}\right) - 75\right)^+, \text{ where } P^{mid} := (P^{ask} + P^{bid})/2.$$

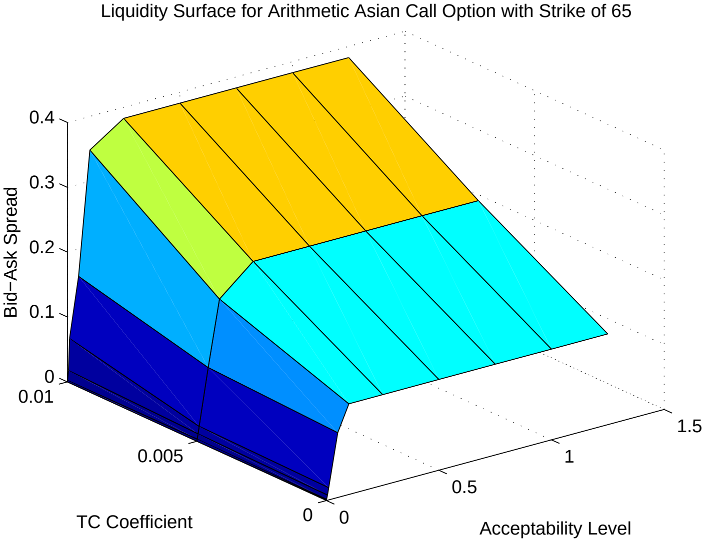
Table: *Prices of an Arithmetic Asian Call Option with* $\lambda = 0$

γ	S_0^{ask}	$\Pi_0^{ask,\gamma}$	$\Pi_0^{bid,\gamma}$	S_0^{bid}
0.0001	1.388854	1.341775	1.341559	1.250035
0.001	–	1.342746	1.340587	–
0.005	–	1.347062	1.336288	–
0.01	–	1.352446	1.330952	–
0.05	–	1.388853	1.289754	–
0.1	–	1.388853	1.250036	–
0.25	–	1.388853	1.250036	–
0.5	–	1.388854	1.250036	–
0.75	–	1.388854	1.250036	–
1	–	1.388854	1.250035	–
1.25	–	1.388854	1.250035	–

Table: *Prices of an Arithmetic Asian Call Option with* $\lambda = 0.005$

γ	S_0^{ask}	$\Pi_0^{ask,\gamma}$	$\Pi_0^{bid,\gamma}$	S_0^{bid}
0.0001	1.484025	1.376819	1.376598	1.230204
0.001	–	1.377816	1.375601	–
0.005	–	1.382244	1.371189	–
0.01	–	1.387769	1.365714	–
0.05	–	1.431586	1.323440	–
0.1	–	1.484025	1.274140	–
0.25	–	1.484024	1.230207	–
0.5	–	1.484024	1.230205	–
0.75	–	1.484024	1.230205	–
1	–	1.484025	1.230205	–
1.25	–	1.484025	1.230205	–

Asian Call Option



Thank You !

The end of the talk ...
but not of the story