PROBABILISTIC APPROACH TO MEAN FIELD GAMES

René Carmona

Department of Operations Research & Financial Engineering
Bendheim Center for Finance
Princeton University

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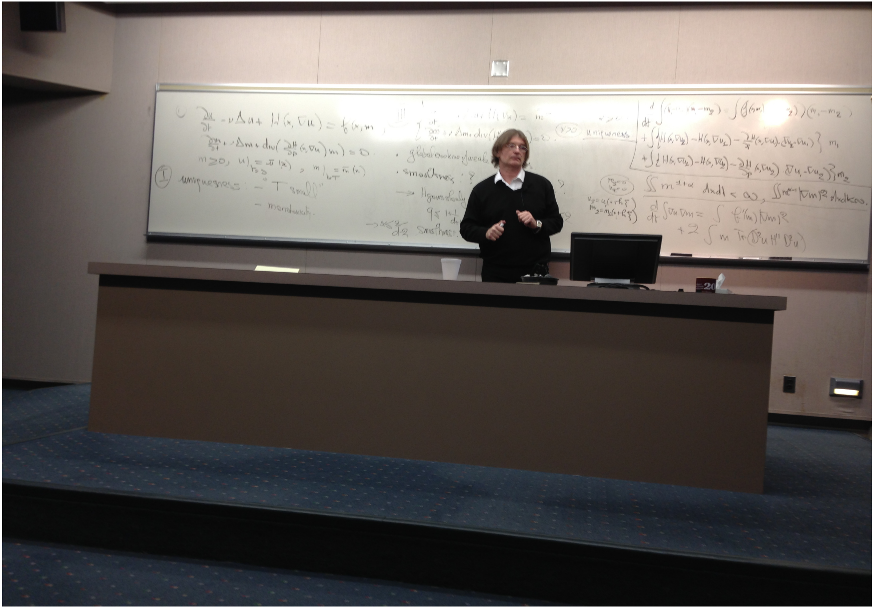
Motivation

Personal Motivation: Trying to Understand



The PDE Approach to MFG in LaTeX

Formulation (given $m(0,\cdot)$ & $u(T,\cdot)$)

$$\partial_t u + \frac{\sigma^2}{2}\Delta u + H(\nabla u) - \rho u = -g(m) \qquad \text{(Hamilton-Jacobi-Bellman)}$$

$$\partial_t m + \nabla\cdot(mH'(\nabla u)) = \frac{\sigma^2}{2}\Delta m, \qquad \text{(Kolmogorov)}$$

where $m(t,\cdot)$ probability measure, $H(p) = \sup_a(ap - h(a))$.

Stationary Case

$$\frac{\sigma^2}{2}\Delta u + H(\nabla u) - \rho u = -g(m)$$

$$\nabla\cdot(mH'(\nabla u)) = \frac{\sigma^2}{2}\Delta m,$$

Stochastic Control Problem followed by a **Fixed Point**

$$u(t,x) = \sup_{(\alpha_s)_{t\le s\le T}, X_t = x} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}[g(m(s,X_s)) + h(|\alpha(s,X_s)|)]ds\right]$$

under constraint $dX_t = \alpha(t,X_t)dt + \sigma dW_t$ (HJB), with $m(t,x)$ density of X_t (Kolmogorov).

Probabilistic Approach

Disclaimer (to PL and the PDE *aficionados*)

> ***"Mathematicians (Probabilists) are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different."***

Johann Wolfgang von Goethe

Probabilistic Approach

- (Pontryagin) Stochastic Maximum Principle
- **FBSDEs** of **McKean Vlasov** type
- **Weak Formulation** and BSDEs
- Control of **McKean-Vlasov** stochastic differential equations

LECTURES BASED ON

- (with F. Delarue and A. Lachapelle) Control of McKean-Vlasov Dynamics versus Mean Field Games. *MAFE* (2012)
- (with F. Delarue) Probabilistic Analysis of Mean Field Games. *SIAM J. Optimization and Contol*
- (with F. Delarue) Control of McKean Vlasov Dynamics *submitted*
- (with F. Delarue) FBSDEs of McKean-Vlasov Type I. Existence *Electronic Communications in Probability*
- (with D. Lacker) The Weak Formulation Approach to Mean Field Games. *submitted*
- (with J.P. Fouque and L.H. Sun) Systemic Risk and Mean Field Games. *submitted*
- Lecture Notes on Stochastic Control and Stochastic Differential Games. *Princeton University*

Not cited in these lectures, the **other sources will be** !

A First Example of Stochastic (Differential) Game

Motivating Toy Model from Systemic Risk

- $X_t^i, i = 1, \dots, N$ log-monetary reserves of N banks
- $B_t^i, i = 1, \dots, N$ **standard Brownian motions**, $\sigma > 0$
- **Borrowing and lending** through the drifts:

$$
\begin{aligned}
dX_t^i &= \frac{a}{N} \sum_{j=1}^{N} (X_t^j - X_t^i)\, dt + \sigma dB_t^i \\
&= a(\overline{X}_t - X_t^i)\, dt + \sigma dB_t^i\,, \quad i = 1, \dots, N.
\end{aligned}
$$

- OU processes reverting to the **sample mean** $\overline{X}_t$ (rate $a > 0$)
- $D < 0$ **default level**

Easy Conclusions

- $\overline{X}_t$ is a BM a Brownian motion with vol. of the order $\sigma/\sqrt{N}$;
- Simulations “show" that **STABILITY** is created by increasing the rate a;
- Easy to compute the loss distribution (how many firms fail);
- Large Deviations (Gaussian estimates) show that increasing a increases **SYSTEMIC RiSK**

A Competitive Equilibrium Analog

- $X_t^i, i = 1, \ldots, N$ log-monetary reserves of N banks
- $W_t^i, i = 0, 1, \ldots, N$ **independent Brownian motions**, $\sigma > 0$
- **Borrowing and lending** through the drifts:

$$dX_t^i = \left[a(\overline{X}_t - X_t^i) + \alpha_t^i\right] dt + \sigma\left(\sqrt{1-\rho^2}dW_t^i + \rho dW_t^0\right), \quad i = 1, \cdots, N$$

α^i **is the control of bank** i which tries to **minimize**

$$J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left\{\int_0^T \left[\frac{1}{2}(\alpha_t^i)^2 - q\alpha_t^i(\overline{X}_t - X_t^i) + \frac{\epsilon}{2}(\overline{X}_t - X_t^i)^2\right] dt + \frac{\epsilon}{2}(\overline{X}_T - X_T^i)^2\right\}$$

Regulator chooses $q > 0$ to control the cost of borrowing and lending.

- If X_t^i is small (relative to the empirical mean $\overline{X}_t$) then bank i will want to borrow($\alpha_t^i > 0$)
- If X_t^i is large then bank i will want to lend ($\alpha_t^i < 0$)

Example of **Mean Field Game (MFG)**

Crash Course on Stochastic Differential Games

State Dynamics

Time evolution of the **state** $\underline{X} = \underline{X}^\alpha$ of the **system**:

$$dX_t = b(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dW_t \qquad 0 \le t \le T,$$

with $X_0 = x$ and where

$$b : [0.T] \times \Omega \times \mathbb{R}^d \times A \hookrightarrow \mathbb{R}^d \qquad \text{and} \qquad \sigma : [0.T] \times \Omega \times \mathbb{R}^d \times A \hookrightarrow \mathbb{R}^{d\times m}$$

satisfy

(A) $(b(t, x, \alpha))_{0\le t\le T}$ and $(\sigma(t, x, \alpha))_{0\le t\le T}$ progressively measurable;
(B) **Lipschitz** coefficients

$$|b(t, \omega, x, \alpha) - b(t, \omega, x', \alpha)| + |\sigma(t, \omega, x, \alpha) - \sigma(t, \omega, x', \alpha)| \le c|x - x'|$$

Most often $X_t = (X_t^1, \cdots, X_t^N)$ and $\alpha_t = (\alpha_t^1, \cdots, \alpha_t^N)$ with

- X_t^i **private state**
- α_t^i **action** (control)

at time t of player $i \in \{1, \cdots, N\}$

ADMISSIBLE STRATEGY PROFILES

$\underline{\alpha} \in \mathbb{A}$ if $\underline{\alpha} = (\alpha_t)_{0 \le t \le T}$ satisfies

- **Integrability Properties**
- **Measurability Properties**
  - **Open Loop** (OL): $\underline{\alpha} = (\alpha_t)_{0 \le t \le T}$ is $\mathcal{F}_t^W$- adapted

$$\alpha_t = \phi(t, W_{[0,t]})$$

  - **Closed Loop** (CL): $\underline{\alpha} = (\alpha_t)_{0 \le t \le T}$ is $\mathcal{F}^{X_{[0,t]}}$- adapted

$$\alpha_t = \phi(t, X_[0, t])$$

  - **Closed Loop in Feedback Form** (CLFF): $\underline{\alpha} = (\alpha_t)_{0 \le t \le T}$ is $\mathcal{F}^{X_t}$- adapted

$$\alpha_t = \phi(t, X_t)$$

    (**Markovian control**)
  - **Distributed Markovian Controls**: $\underline{\alpha}^i = (\alpha_t^i)_{0 \le t \le T}$ is $\mathcal{F}^{X_t^i}$- adapted

$$\alpha_t^i = \phi^i(t, X_t^i)$$

  - ....

Cost Functions

- (**Terminal Cost**) a $\mathcal{F}_T$-measurable r.v. $\xi^i \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$
  Most often, $\xi^i = g^i(X_T)$ for some $g^i : \Omega \times \mathbb{R}^d \hookrightarrow \mathbb{R}$;
- (**Running Cost**) $f^i : [0, T] \times \Omega \times \mathbb{R}^d \times A \hookrightarrow \mathbb{R}$ (same assumption as the drift b);

- (**Cost Functional**) If the N players use the strategy profile $\alpha \in \mathbb{A}$, the expected total cost to player i is

$$J^i(\alpha) = \mathbb{E}[\int_0^T f^i(s, X_s, \alpha_s)ds + \xi^i\}, \qquad \underline{\alpha} = (\underline{\alpha}^1, \cdots, \underline{\alpha}^N) \in \mathbb{A}. \tag{1}$$

Pareto Optimality

Players try to minimize

$$J(\underline{\alpha}) = (J^1(\underline{\alpha}), \cdots, J^N(\underline{\alpha})), \qquad \underline{\alpha} \in \mathbb{A}$$

Definition

An admissible strategy profile $\underline{\alpha}^* = (\underline{\alpha}^{*1}, \cdots, \underline{\alpha}^{*N}) \in \mathbb{A}$ is said to be **Pareto optimal** if there is **no** $\underline{\alpha} = (\underline{\alpha}_1, \cdots, \underline{\alpha}_N) \in \mathbb{A}$ s.t.

$$\forall i \in \{1, \cdots, N\}, \quad J^i(\underline{\alpha}) \leq J^i(\underline{\alpha}^*)$$
$$\exists i_0 \in \{1, \cdots, N\}, \quad J^{i_0}(\underline{\alpha}) < J^{i_0}(\underline{\alpha}^*).$$

I.e., there is no strategy which makes ***every player*** at least as well off and ***at least one player*** strictly better off.

Natural in problems of **optimal allocation of resources** (economics, operations research)

Notions of Nash Equilibrium

Definition

(GENERIC) A set of admissible strategies $\underline{\alpha}^* = (\underline{\alpha}^{*1}, \cdots, \underline{\alpha}^{*N}) \in \mathbb{A}$ is said to be a Nash equilibrium for the game if

$$\forall i \in \{1, \cdots, N\}, \forall \underline{\alpha}^i \in \mathbb{A}^i, \qquad J^i(\underline{\alpha}^*) \leq J^i(\underline{\alpha}^{*-i}, \underline{\alpha}^i).$$

No single player can be better off by perturbing **unilaterally** his strategy

Will be **refined** and specialized to different **information structures**

Search for Nash Equilibriums

- Construction of **Best Response Map**
  - for each **strategy profiles** $(\alpha^1, \cdots, \alpha^N)$
  - for each $i \in \{1, \cdots, N\}$
  - find $\hat{\alpha}^i$ minimizing $J^i(\alpha^1, \cdots, \alpha^N)$ over α^i
  - $(\alpha^1, \cdots, \alpha^N) \hookrightarrow (\hat{\alpha}^1, \cdots, \hat{\alpha}^N)$

- Find a **fixed point** for the Best Response map

Can be quite **involved** (**prohibitive** when N is large)

- Typically very difficult to prove existence
- Most often no uniqueness
- Numerical computations very difficult (especially when N is large)

Markov Equilibriums

Strategy profiles in **Closed Loop Feedback Form**.
In the Markovian case, we assume that

the coefficients b and σ are Lipschitz in (x, α) uniformly in $t \in [0, T]$

$\phi = (\varphi^1, \cdots, \varphi^N)$ with **deterministic** functions $\varphi^i : [0, T] \times \mathbb{R}^d \hookrightarrow \mathbb{R}^k$ is a **Markov Nash equilibrium** (MNE), if **for each** $(t, x) \in [0, T] \times \mathbb{R}^d$, $\underline{\alpha}^* = (\underline{\alpha}^{*1}, \cdots, \underline{\alpha}^{*N}) \in \mathbb{A}$ defined by

$$\alpha_s^{*i} = \varphi(t, X_s^{t,x}), \qquad s \in [t, T]$$

where $\underline{X}^{t,x}$ is the unique solution of the stochastic differential equation

$$dX_s = b(s, X_s, \phi(s, X_s))ds + \sigma(s, X_s, \phi(s, X_s))dW_s, \qquad t \le s \le T$$

with initial condition $X_t = x$, satisfies the usual definition inequalities

- The **same** ϕ solves the game on **ALL** $[t, T]$ for **ALL** initial conditions $X_t = x$;
- **sub game perfect**

PDE Formulation

V^i the ***value function*** of player i:

$$(t,x) \hookrightarrow V^i(t,x) = \inf_{\underline{\alpha}^i \in \mathbb{A}^i} \mathbb{E}\{\int_0^T f^i(t, X_t, (\alpha^{*-i}(t,X_t), \alpha^i_t))dt + g_i(X_T)\}$$

expected to satisfy the ***HJB equation***

$$\partial_t V^i + L^{*i}(x, \partial_x V^i(t,x), \partial^2_{xx} V^i(t,x)) = 0 \qquad (2)$$

where $L^{*i}(x,y,z) \inf_{\alpha \in A^i} L^i(x,y,z,\alpha)$ with

$$L^i(x,y,z,\alpha) = \frac{1}{2}\text{trace}\Big[z[\sigma\sigma^\dagger](t,x,(\alpha^{*-i}(t,x),\alpha))\Big] - y \cdot b(t,x,(\alpha^{*-i}(t,x),\alpha)) + f^i(t,x,(\alpha^{*-i}(t,x),\alpha))$$

- System of ***coupled*** HJB equations
- Usually very difficult to solve (existence & uniqueness)
- In many examples below that $\alpha^{*j}(t,x) = \partial_x V^j(t,x)$

Mean Field Interactions

Idea from **statistical physics**

- **Interactions** between **palyers' states**
  - in the coefficients of the **state dynamics**
  - in the **cost functions**
- exclusively through the **empirical distribution**

$$\mu_t^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{X_t^j}$$

Consequences:

- Strong **symmetry** among the players
- Each player can **hardly influence** the system when N is large.

Examples of Mean Field Interactions

Scalar Interactions

$$b(t, x, \mu, \alpha) = \tilde{b}(t, x, \langle \psi, \mu \rangle, \alpha) \qquad \sigma(t, x, \mu, \alpha) = \sigma$$

so that

$$dX_t^i = \tilde{b}\Big(t, X_t^i, \frac{1}{N}\sum_{j=1}^N \psi(X_t^j), \alpha_t^i\Big)dt + \sigma dW_t^i$$

Linear interactions, *of order* 1

$$b(t, x, \mu, \alpha) = \int \tilde{b}(t, x, x', \alpha)\, d\mu(x')$$

so

$$\begin{aligned} dX_t^i &= b(t, X_t^i, \overline{\mu}_t^N, \alpha_t^i)dt + \sigma dW_t^i \\ &= \frac{1}{N}\sum_{j=1}^N \tilde{b}(t, X_t^i, X_t^j, \alpha_t^i)dt + \sigma dW_t^i \end{aligned}$$

Similar forms for the cost functions

Approximate Nash Equilibriums

The strategies $(\alpha_t^{N,i})_{i=1,\cdots,N}$ form an

ϵ-approximate Nash equilbrium

for the N-player game if for $1 \leq i \leq N$ and $\beta \in \mathbb{A}^i$,

$$J^{N,i}(\alpha^{N,1},\dots,\alpha^{N,i-1},\beta,\alpha^{N,i+1},\dots,\alpha^{N,N}) \leq J^{N,i}(\alpha^{N,1},\dots,\alpha^{N,N}) + \epsilon.$$

For large games ($N \to \infty$) we look for a sequence $(\epsilon_N)_{N\geq 0}$ and an ϵ_N-approximate Nash equilbrium with

$$\lim_{N\to\infty} \epsilon_N = 0$$

Pontryagin Stochastic Maximum Principle

Players' Hamiltonians

for each player $i \in \{1, \cdots, N\}$, we define his Hamiltonian as the function H^i:

$$[0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d \times m} \times A \ni (t, x, y, z, \alpha) \hookrightarrow H^i(t, x, y, z, \alpha) \in \mathbb{R}$$

defined by

$$H^i(t, x, y, z, \alpha) = \underbrace{b(t, x, \alpha) \cdot y}_{\text{inner product of state drift } b \text{ and covariable } y} + \underbrace{\text{trace}\,[\sigma(t, x, \alpha)^\dagger z]}_{\text{inner product of state volatility } \sigma \text{ and covariable } z} + \underbrace{f^i(t, x, \alpha)}_{\text{running cost of player } i}$$

Adjoint Equations & Adjoint Processes

Given

- an open loop admissible strategy profile $\underline{\alpha} \in \mathbb{A}$
- the corresponding evolution $\underline{X} = \underline{X}^{\alpha}$ of the state of the system,

a set of N couples $(\underline{Y}^{i,\alpha}, \underline{Z}^{i,\alpha}) = (Y_t^{i,\alpha}, Z_t^{i,\alpha})_{t\in[0,T]}$ of processes is said to be a set of **adjoint processes** associated with $\underline{\alpha} \in \mathbb{A}$ if

$$\begin{cases} dY_t^{i,\alpha} = -\partial_x H^i(t, X_t, Y_t^{i,\alpha}, Z_t^{i,\alpha}, \alpha_t)dt + Z_t^{i,\alpha} dW_t \\ Y_T^{i,\alpha} = -\partial_x g^i(X_T^{\alpha}). \end{cases}$$

Existence and uniqueness **easy** from classical BSDE theory

Pontryagin SMP: Necessary Conditions

Under the above conditions, if

- $\alpha^* \in \mathbb{A}$ is an open loop Nash equilibrium,
- $X^* = (X_t^*)_{0 \le t \le T}$ is the corresponding controlled state of the system
- $(\underline{Y}^*, \underline{Z}^*) = \big((\underline{Y}^{*1}, \cdots, \underline{Y}^{*N}), (\underline{Z}^{*1}, \cdots, \hat{Z}^{*N})\big)$ are the adjoint processes

then the generalized min-max **Isaacs conditions** hold **along the optimal paths**:

$$H^i(t, X_t^*, Y_t^{*i}, Z_t^{*i}, \alpha_t^*) = \inf_{\alpha^i \in A^i} H^i(t, X_t^*, Y_t^{*i}, Z_t^{*i}, (\alpha^{*-i}, \alpha^i)), \quad dt \otimes d\mathbb{P} \text{ a.s.;}$$

for $i \in \{1, \cdots, N\}$

Isaacs Conditions

We say that the generalized **Isaacs** (minmax) **conditions** hold if there exists a function

$$\hat{\alpha} : [0, T] \times \mathbb{R}^d \times (\mathbb{R}^d)^N \times (\mathbb{R}^{d\times m})^N \ni (t, x, y, z) \hookrightarrow \hat{\alpha}(t, x, y, z) \in A$$

satisfying

$$H^i(t, x, y^i, z^i, \hat{\alpha}(t, x, y, z)) \leq H^i(t, x, y^i, z^i, (\hat{\alpha}(t, x, y, z)^{-i}, \alpha^i))$$

for all $\alpha^i \in A^i$, $i \in \{1, \cdots, N\}$, $t \in [0, T]$, $x \in \mathbb{R}^d$,
$y = (y^1, \cdots, y^N) \in (\mathbb{R}^d)^N$, and $z = (z^1, \cdots, z^N) \in (\mathbb{R}^{d\times m})^N$.

Pontryagin SMP: Sufficient Conditions

Assume

- Coefficients **twice continuously differentiable** in $(x, \alpha) \in \mathbb{R}^d \times A$
- **Bounded** partial derivatives,
- $\hat{\alpha} \in \mathbb{A}$ is an admissible adapted (open loop) strategy profile,
- $\underline{\hat{X}} = (\hat{X}_t)_{0 \le t \le T}$ the corresponding controlled state,
- $(\underline{\hat{Y}}, \underline{\hat{Z}}) = ((\underline{\hat{Y}}^1, \cdots, \underline{\hat{Y}}^N), (\underline{\hat{Z}}^1, \cdots, \underline{\hat{Z}}^N))$ adjoint processes,

if **FURTHERMORE**t for each $i \in \{1, \cdots, N\}$:

1. $(x, \alpha) \hookrightarrow H^i(t, x, \hat{Y}^i_t, \hat{Z}^i_t, \alpha)$ is a **convex** function , $dt \otimes d\mathbb{P}$ a.s.;
2. g^i is **convex** $\mathbb{P}$-a.s.
3. $H^i(t, \hat{X}_t, \hat{Y}^i_t, \hat{Z}^i_t, \hat{\alpha}_t) = \inf_{\alpha^i \in A^i} H^i(t, \hat{X}_t, \hat{Y}^i_t, \hat{Z}^i_t, (\hat{\alpha}^{-i}, \alpha^i))$, $dt \otimes d\mathbb{P}$ a.s.

then $\underline{\hat{\alpha}}$ is an open loop Nash equilibrium.

Implementation Strategy

If above assumptions are satisfied

1. search for a deterministic function

   $$[0,T]\times\mathbb{R}^d\times\mathbb{R}^{dN}\times\mathbb{R}^{dmN}\ni(t,x,(y^1,\cdots,y^N),(z^1,\cdots,z^N))\hookrightarrow\hat{\alpha}(t,x,(y^1,\cdots,y^N),(z^1,\cdot$$

   satisfying Isaacs conditions;

2. replace the *adapted* controls $\underline{\alpha}$ in the **forward dynamics** of the state **AND** in the **adjoint BSDEs** by

   $$\hat{\alpha}(t,X_t,(Y_t^1,\cdots,Y_t^N),(Z_t^1,\cdots,Z_t^N))$$

3. solve the **large strongly coupled FBSDE** system:

$$\begin{cases} dX_t = b(t,X_t,\hat{\alpha}(t,X_t,(Y_t^1,\cdots,Y_t^N),(Z_t^1,\cdots,Z_t^N)))dt+\sigma(t,X_t,\hat{\alpha}(\cdots\cdots))dW_t,\\ dY_t^1 = -\partial_x H^1(t,X_t,Y_t^1,Z_t^1,\hat{\alpha}(t,X_t,(Y_t^1,\cdots,Y_t^N),(Z_t^1,\cdots,Z_t^N)))dt+Z_t^1 dW_t,\\ \cdots\cdots=\cdots\\ dY_t^N = -\partial_x H^N(t,X_t,Y_t^N,Z_t^N,\hat{\alpha}(t,X_t,(Y_t^1,\cdots,Y_t^N),(Z_t^1,\cdots,Z_t^N)))dt+Z_t^N dW_t, \end{cases}$$

   with $X_0=x$ and $Y_T^i=\partial_x g^i(X_T)$

4. if successful, $\hat{\alpha}_t=\hat{\alpha}(t,X_t,(Y_t^1,\cdots,Y_t^N),(Z_t^1,\cdots,Z_t^N))$ **is an open loop Nash equilibrium !**

Folk Wisdom

If you consider **open loop game model**

use Pontryagin stochastic maximum principle

and reduce the problem to

1. finding a function satisfying Isaacs conditions;
2. solving a coupled FBSDE system

iIf you consider **Markov game model**

use PDE approach based on system of coupled HJB equations

and reduce the problem to

1. solving scalar optimizations (akin to Isaacs conditions);
2. solving a coupled system of nonlinear (HJB) PDEs

Personal opinion: **NOT ALWAYS the best strategy**

Adjoint Processes in Markovian Games

Assume

- $\phi = (\varphi^1, \cdots, \varphi^N)$ is jointly measurable function from $[0, T] \times \mathbb{R}^d$ into $A = A^1 \times \cdots \times A^N$
- ϕ differentiable in x with derivatives uniformly bounded in (t, x)
- b and σ are Lipschitz in (x, α) uniformly in $t \in [0, T]$,

$\underline{X}^\phi$ the unique strong solution of the state equation:

$$dX_t = b(t, X_t, \phi(t, X_t))dt + \sigma(t, X_t, \phi(t, X_t))dW_t, \qquad X_0 = x.$$

$(\underline{Y}^{\phi,i}, \underline{Z}^{\phi,i}) = (Y_t^{\phi,i}, Z_t^{\phi,i})_{t\in[0,T]}$ **adjoint processes** associated with ϕ if

$$\begin{cases} dY_t^{\phi,i} = -[\partial_x H^i(t, X_t^\phi, Y_t^{\phi,i}, Z_t^{\phi,i}, \phi(t, X_t)) \\ \qquad\qquad + \sum_{j=1, j\neq i}^N \partial_{\alpha^j} H^i(t, X_t^\phi, Y_t^{\phi,i}, Z_t^{\phi,i}, \phi(t, X_t))\partial_x \varphi^j(t, X_t^\phi)]dt + Z_t^{\phi,i} dW_t \\ Y_T^{\phi,i} = -\partial_x g^i(X_T^\phi). \end{cases}$$

Again existence and uniqueness of the adjoint processes from classical BSDE theory.

Stochastic Maximum Principle for MNEs

Assume

- Coefficients **twice continuously differentiable** in $(x, \alpha) \in \mathbb{R}^d \times A$
- **Bounded** partial derivatives,
- $\phi = (\varphi^1, \cdots, \varphi^N)$ is continuously differentiable in $x \in \mathbb{R}^d$ for $t \in]0, T]$ fixed, with bounded partial derivatives,
- $\underline{X}^\phi = (X_t^\phi)_{0 \le t \le T}$ the corresponding controlled state,
- $(\underline{Y}^\phi, \underline{Z}^\phi) = ((\underline{Y}^{\phi,1}, \cdots, \underline{Y}^{\phi,N}), (\underline{Z}^{\phi,1}, \cdots, \underline{Z}^{\phi,N}))$ adjoint processes of ϕ,

if **FURTHERMORE**t for each $i \in \{1, \cdots, N\}$:

1. $(x, \alpha) \hookrightarrow H^i(t, x, Y_t^{\phi,i}, Z_t^{\phi,i}, \alpha)$ is **convex** , $\quad dt \otimes d\mathbb{P}$ a.s.;
2. g^i is **convex** $\mathbb{P}$-a.s.
3. $H^i(t, X_t^\phi, Y_t^{\phi,i}, Z_t^{\phi,i}, \phi(t, X_t^\phi)) =$ $\inf_{\alpha^i \in A^i} H^i(t, X_t^\phi, Y_t^{\phi,i}, Z_t^{\phi,i}, (\phi(t, X_t^\phi)^{-i}, \alpha^i))$, $dt \otimes d\mathbb{P}$ a.s.

then ϕ is a **Markov Nash equilibrium** (MNE).

Complete Analysis of the Systemic Risk Toy Model

Solving for an Open Loop Nash Equilibrium

For each player $i \in \{1, \cdots, N\}$,

- $\mathbb{H}^2$ space of admissible strategies
  (square integrable adapted processes)
- Hamitonian of player i reads:

$$\tilde{H}^i(x, y, \alpha) = \sum_{j=1}^{N}[a(\overline{x} - x^j) + \alpha^j]y^j + \frac{1}{2}(\alpha^i)2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2$$

- Minimized by

$$\hat{\alpha}^i = \hat{\alpha}^i(x, y) = -y^i + q(\overline{x} - x^i).$$

Probabilistic Approach

Adjoint Equations

- Given an admissible strategy profile $\underline{\alpha} = (\underline{\alpha}^1, \cdots, \underline{\alpha}^N)$
- The corresponding controlled state $X_t = X_t^{\underline{\alpha}}$,
- The **adjoint processes** associated to $\underline{\alpha}$ are the processes $(\underline{Y}, \underline{Z}) = ((\underline{Y}^1, \cdots, \underline{Y}^N), (\underline{Z}^1, \cdots, \underline{Z}^N))$ solving the system of BSDEs:

$$dY_t^{i,j} = -\partial_{x^j} \tilde{H}^i(X_t, Y_t^i, \alpha_t)dt + \sum_{k=0}^{N} Z_t^{i,k} dW_t^k,$$

$$= -\left[\sum_{k=1}^{N} a(\frac{1}{N} - \delta_{k,j}) Y_t^{i,k} - q\alpha_t^i(\frac{1}{N} - \delta_{i,j}) + \epsilon(\overline{X}_t - X_t^i)(\frac{1}{N} - \delta_{i,j})\right] dt + \sum_{k=0}^{N} Z_t^{i,j,k} dW_t^k.$$

with $Y_T^i = c(\overline{X}_T - X_T^i)(\frac{1}{N} - 1)$ for $i, j = 1, \cdots, N$.

Strategy:

- **Replace** all the occurrences of the controls α_t^i in the **forward** and **backward** (adjoint) equations by

$$\hat{\alpha}^i(X_t, Y_t^i) = -Y_t^{i,i} + q(\overline{X}_t - X_t^i)$$

- **Solve** the resulting system of (coupled) **FBSDEs**
- once done, $\alpha_t^i = \hat{\alpha}^i(X_t, Y_t^i) = -Y_t^{i,i} + q(\overline{X}_t - X_t^i)$ form an **open loop Nash equilibrium**.

Pontryagin Maximumn Principle Approach (cont.)

The FBSDEs read

$$\begin{cases} dX_t^i = [(a+q)(\overline{X}_t - X_t^i) - Y_t^{i,i}]dt + \sigma\rho dW_t^0 + \sigma\sqrt{1-\rho^2}dW_t^i, \qquad i = 1,\cdots,N \\ dY_t^{i,j} = -\Big[a\sum_{k=1}^N(\frac{1}{N} - \delta_{k,j})Y_t^{i,k} - q[Y_t^{i,i} - q(\overline{X}_t - X_t^i)](\frac{1}{N} - \delta_{i,j}) \\ \qquad\qquad\qquad + \epsilon(\overline{X}_t - X_t^i)(\frac{1}{N} - \delta_{i,j}) + \sum_{k=0}^N Z_t^{i,j,k}dW_t^k \\ Y_T^{i,j} = c(\overline{X}_T - X_T^i)(\frac{1}{N} - \delta_{i,j}) \qquad\qquad i,j = 1,\ldots,N. \end{cases}$$

Affine FBSDE, so we look for a solution $Y_t = P_t X_t + p_t$. Since the couplings depend only upon quantities of the form $\overline{X}_t - X_t^i$

$$Y_t^{i,j} = \eta_t(\overline{X}_t - X_t^i)(\frac{1}{N} - \delta_{i,j})$$

for some deterministic function $t \hookrightarrow \eta_t$ to be determined.

Simple Derivations

Computing the differential $dY_t^{i,j}$ we get

$$dY_t^{i,j} = \left(\frac{1}{N} - \delta_{i,j}\right)(\overline{X}_t - X_t^i)\left[\dot{\eta}_t - \eta_t\left(a + q + (1 - \frac{1}{N})\eta_t\right)\right]$$
$$+ \sigma\sqrt{1-\rho^2}\eta_t(\frac{1}{N} - \delta_{i,j})\left(\frac{1}{N}\sum_{k=1}^{N} dW_t^k - dW_t^i\right).$$

Evaluating the RHS of the BSDE using the ansatz for $Y_t^{i,j}$ we get

$$dY_t^{i,j} = -\left[a\sum_{k=1}^{N}(\frac{1}{N} - \delta_{k,j})[\eta_t(\overline{X}_t - X_t^i)(\frac{1}{N} - \delta_{i,k})] + \epsilon(\overline{X}_t - X_t^i)(\frac{1}{N} - \delta_{i,j})\right.$$
$$\left. - q[\eta_t(\overline{X}_t - X_t^i)(\frac{1}{N} - 1) - q(\overline{X}_t - X_t^i)](\frac{1}{N} - \delta_{i,j}) + \sum_{k=0}^{N} Z_t^{i,j,k} dW_t^k\right.$$
$$= \left(\frac{1}{N} - \delta_{i,j}\right)(\overline{X}_t - X_t^i)\left[(a+q)\eta_t - \frac{1}{N}(\frac{1}{N} - 1)\eta_t^2 + q^2 - \epsilon\right]dt + \sum_{k=0}^{N} Z_t^{i,j,k} dW_t^k.$$

The Unavoidable Riccati Equation

Identifying the two Itô decompositions of $Y_t^{i,j}$ we get:

$$Z_t^{i,j,0} = 0, \quad Z_t^{i,j,k} = \sigma\sqrt{1-\rho^2}\eta_t(\frac{1}{N} - \delta_{i,j})(\frac{1}{N} - \delta_{i,k}), \quad k = 1, \cdots, N$$

and

$$\dot{\eta}_t - \eta_t\left(a + q + (1 - \frac{1}{N})\eta_t\right) = (a+q)\eta_t - \frac{1}{N}(\frac{1}{N} - 1)\eta_t^2 + q^2 - \epsilon$$

which we rewrite as a standard **scalar** Riccati's equation

$$\dot{\eta}_t = 2(a+q)\eta_t + (1 - \frac{1}{N^2})\eta_t^2 + q^2 - \epsilon$$

with terminal condition $\eta_T = c$. Under the condition $\epsilon \geq q^2$ (which guarantees the **convexity** of the running cost function f^i), this Riccati equation admits a **unique solution**.

A Couple of Noteworthy Remarks

- Since

$$\alpha_t^i = [q - \eta_t(\frac{1}{N} - 1)](\overline{X}_t - X_t^i)$$

the equilibrium controls are in **closed loop feedback form** (i.e. depend only upon X_t at time t). However,

They do not form a closed loop Nash equilibrium !!!!!

- In equilibrium, the dynamics of X_t are given by

$$dX_t^i = [a + q - \eta_t(\frac{1}{N} - 1)](\overline{X}_t - X_t^i)dt + \sigma\rho dW_t^0 + \sigma\sqrt{1-\rho^2}dW_t^i,$$

$i = 1, \cdots, N$, OUs with mean reversion rate

a replaced by $a + q - \eta_t(\frac{1}{N} - 1)$.

Solving for a Closed Loop Nash Equilibrium

Still by the Stochastic Maximum Approach

- Search for a set $\phi = (\varphi^1, \cdots, \varphi^N)$ of **feedback functions** φ^i
- The Hamitonian of player $i \in \{1, \cdots, N\}$ reads:

$$H^{-i}(x, y, \alpha) = \sum_{k=1, k\neq i}^{N} [a(\overline{x}-x^k)+\varphi^k(t,x)]y^k+[a(\overline{x}-x^i)+\alpha]y^i+\frac{1}{2}\alpha^2-q\alpha(\overline{x}-x^i)+\frac{\epsilon}{2}(\overline{x}-x^i)^2$$

- The value of α minimizing this Hamiltonian is the same as before
- For the same reasons as before, we make the **ansatz**

$$\varphi^i(t,x) = [q-\eta_t(\frac{1}{N}-1)](\overline{x}-x^i), \qquad (t,x) \in [0,T]\times\mathbb{R}^d,\ i=1,\cdots,N,$$

for some deterministic function $t \hookrightarrow \eta_t$

Constructing the Best Response by Pontryagin

Solving the FBSDE

$$\begin{cases} dX_t^i = [(a+q)(\overline{X}_t - X_t^i) - Y_t^{i,i}]dt + \sigma\rho dW_t^0 + \sigma\sqrt{1-\rho^2}dW_t^i, \qquad i = 1,\cdots,N \\ dY_t^{i,j} = -\Big[a\sum_{k=1}^N(\frac{1}{N} - \delta_{k,j})Y_t^{i,k} + a\sum_{k=1,k\neq i}^N \partial_{x^j}\varphi^k(t,X_t)Y_t^{i,k} \\ \qquad\qquad - q[Y_t^{i,i} - q(\overline{X}_t - X_t^i)](\frac{1}{N} - \delta_{i,j}) + \epsilon(\overline{X}_t - X_t^i)(\frac{1}{N} - \delta_{i,j}) + \sum_{k=0}^N Z_t^{i,j,k}dW_t^k \\ Y_T^{i,j} = c(\overline{X}_T - X_T^i)(\frac{1}{N} - \delta_{i,j}) \qquad i,j = 1,\ldots,N. \end{cases}$$

For the particular choice of feedback functions (ansatz), we have

$$\partial_{x^j}\varphi^k(t,x) = (\frac{1}{N} - \delta_{j,k})[q - \eta_t(\frac{1}{N} - 1)],$$

and the backward component of the BSDE rewrites:

$$dY_t^{i,j} = -\Big[a\sum_{k=1}^N(\frac{1}{N} - \delta_{k,j})Y_t^{i,k} + a\sum_{k=1,k\neq i}^N(\frac{1}{N} - \delta_{j,k})[q - \eta_t(\frac{1}{N} - 1)]Y_t^{i,k}$$

$$- q[Y_t^{i,i} - q(\overline{X}_t - X_t^i)](\frac{1}{N} - \delta_{i,j}) + \epsilon(\overline{X}_t - X_t^i)(\frac{1}{N} - \delta_{i,j}) + \sum_{k=0}^N Z_t^{i,j,k}dW_t^k.$$

Constructing the Best Response by Pontryagin

- For the same reasons as before, we make the **same ansatz** for $Y_t^{i,j}$

$$
\begin{aligned}
dY_t^{i,j} = -\Big[& a\sum_{k=1}^{N}(\frac{1}{N}-\delta_{k,j})\eta_t(\overline{X}_t - X_t^i)(\frac{1}{N}-\delta_{k,j}) \\
& + a\sum_{k=1,k\neq i}^{N}(\frac{1}{N}-\delta_{j,k})[q-\eta_t(\frac{1}{N}-1)]\eta_t(\overline{X}_t - X_t^i)(\frac{1}{N}-\delta_{k,j}) \\
& - q[\eta_t(\overline{X}_t - X_t^i)(\frac{1}{N}-\delta_{i,j}) - q(\overline{X}_t - X_t^i)](\frac{1}{N}-\delta_{i,j}) + \epsilon(\overline{X}_t - X_t^i)(\frac{1}{N}-\delta_{i,j}) \\
& + \sum_{k=0}^{N} Z_t^{i,j,k} dW_t^k. \\
= & \left(\frac{1}{N}-\delta_{i,j}\right)(\overline{X}_t - X_t^i)\left[(a+q-\frac{q}{N})\eta_t + q^2 - \epsilon\right]dt + \sum_{k=0}^{N} Z_t^{i,j,k} dW_t^k.
\end{aligned}
$$

Again, the Unavoidable Riccati Equation

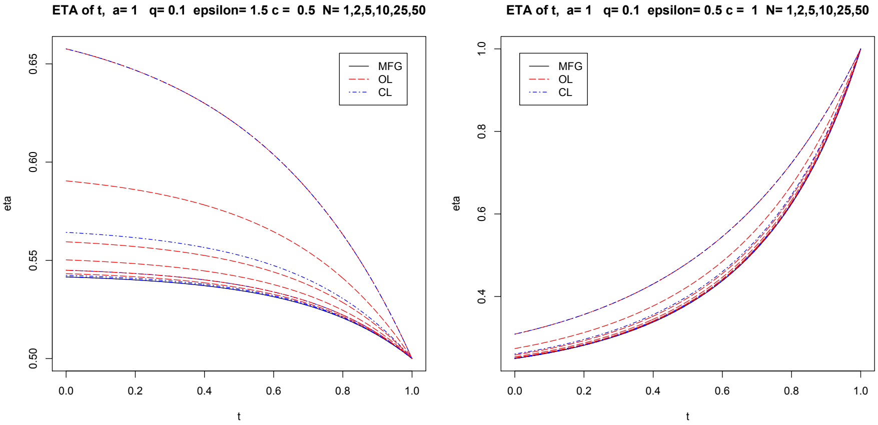
Equating with the differential $dY_t^{i,j}$ from the ansatz, we get the same identification for the $Z_t^{i,j,k}$ as before and the following Riccati equation for η_t:

$$\dot{\eta}_t = 2(a+q)\eta_t + (1 - \frac{1}{N})\eta_t^2 + q^2 - \epsilon$$

with the same terminal condition $\eta_T = c$. We solve this equation under the same condition $\epsilon \geq q^2$.

SAME equation as before with $\frac{1}{N}$ instead of $\frac{1}{N^2}$!!!!

Comparing the Different $t \hookrightarrow \eta(t)$



Plot of the solution η_t of the Riccati equations.

COMMENTS

- In equilibrium, the dynamics of the state X_t are given by:

$$dX_t^i = [a + q - \eta_t(\frac{1}{N} - 1)](\overline{X}_t - X_t^i)dt + \sigma\rho dW_t^0 + \sigma\sqrt{1-\rho^2}dW_t^i,$$

  OUs with mean reversion coefficient a replaced by $a + q - \eta_t(\frac{1}{N} - 1)$.
- The differences between open and closed loop solutions disappear in the limit $N \to \infty$ as they converge toward the **same limit**;
- Both $t \hookrightarrow \eta(t)$ converge toward the solution of the Riccati equation

$$\dot{\eta}_t = 2(a + q)\eta_t + \eta_t^2 + q^2 - \epsilon$$

- This common limit appears as the limit of independent (identical) **classical stochastic control problems** modulo a fixed point (like in the solution of **McKean-Vlasov** stochastic equations)
- The theory of **PROPAGATION OF CHAOS** can be used to construct **approximate Nash equilibriums** with **distributed controls** $\alpha_t^i = \phi(t, X_t^i)$!

More Examples of Mean Field Games

Stoch. Diff. Games with Mean Field Interactions

Player $i \in \{1, \cdots, N\}$ **state process**

$$dX_t^i = b\left(t, X_t^i, \overline{\mu}_t^N, \alpha_t^i\right) dt + \sigma\left(t, X_t^i, \overline{\mu}_t^N, \alpha_t^i\right) dW_t^i,$$

Objective function

$$J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left[\int_0^T f(t, X_t^i, \overline{\mu}_t^N, \overline{\nu}_t^N, \alpha_t^i) dt + g(X_T^i, \overline{\mu}_T^N)\right],$$

where

$$\overline{\mu}_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}, \quad \overline{\nu}_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{\alpha_t^j}$$

EXAMPLE II: A SIMPLE MODEL OF PRICE IMPACT

(**Almgren-Chriss** '01, **Carlin et al.** '09)

- N brokers trade in the same asset and maximize wealth;
- Brokers ($i = 1, \cdots, N$) face identical limit order books;
- Broker i trade at *rate* α_t^i at time t
- **Transaction** price = martingale + drift (**price impact**).

Case of Flat Order Book (Quadratic Costs)

- **Asset price**:

$$dS_t = \frac{\gamma}{N} \sum_{i=1}^{N} \alpha_t^i dt + \sigma_0 dB_t$$

- Broker i's **cash** and **volume**:

$$dK_t^i = -(\alpha_t^i S_t + (\alpha_t^i)^2) dt$$
$$dX_t^i = \alpha_t^i dt + \sigma dW_t^i$$

- Broker i's **wealth**: $V_t^i = V_0^i + X_t^i S_t + K_t^i$,

$$dV_t^i = \left(\frac{\gamma}{N} \sum_{j=1}^{N} \alpha_t^j X_t^i - (\alpha_t^i)^2 \right) dt + \sigma S_t dW_t^i + \sigma_0 X_t^i dB_t$$

Risk Neutral Agents

Broker i **maximizes expected wealth** $\mathbb{E}[V_T^i]$:

$$\sup_{\alpha^i} \mathbb{E} \int_0^T \left(\frac{\gamma}{N} \sum_{j=1}^N \alpha_t^j X_t^i - (\alpha_t^i)^2 \right) dt,$$

$$\text{s.t. } dX_t^i = \alpha_t^i dt + \sigma dW_t^i$$

Are there Nash equilibria?

L-Q Mean Field Game

More General Order Books

- Given a **transaction cost curve** $c : \mathbb{R} \to [0, \infty]$ (convex, $c(0) = 0$);
- **Order book shape function** given by Legendre transform γ;
- **Price impact** given by c':
- Optimization of **expected terminal wealth** becomes:

$$\sup_{\alpha^i} \mathbb{E} \int_0^T \left(\frac{\gamma}{N} \sum_{j=1}^N c'(\alpha_t^j) X_t^i - c(\alpha_t^i) \right) dt,$$

$$\text{s.t. } dX_t^i = \alpha_t^i dt + \sigma dW_t^i$$

In General

- Adding **benchmark tracking penalties**, **carrying and inventory costs**, . . .

$$\sup_{\alpha^i} \mathbb{E}\left[G(X_T^i) + \int_0^T \left(\frac{\gamma}{N} \sum_{j=1}^N c'(\alpha_t^j) X_t^i - c(\alpha_t^i) - F(t, X_t^i) \right) dt \right],$$

s.t. $dX_t^i = \alpha_t^i dt + \sigma dW_t^i$

- Still **MFG** but
  - Brokers' optimization problems **coupled through the empirical distribution of the controls**;
  - Maximizing utility instead of wealth leads to a much harder problem (**common noise would not go away!**)

Example III: A Model of "Flocking"

Deterministic dynamical system model (**Cucker-Smale**)

$$\begin{cases} dx_t^i &= v_t^i dt \\ dv_t^i &= \frac{1}{N}\sum_{j=1}^N w_{i,j}(t)[v_t^i - v_t^j]dt \end{cases}$$

for the weights

$$w_{i,j}(t) = w(|x_t^i - x_t^j|) = \frac{1}{(1 + |x_t^i - x_t^j|^2)^\beta}$$

for some $K > 0$ and $\beta \geq 0$.

If N fixed, $0 \leq \beta \leq 1/2$

- $\lim_{t\to\infty} v_t^i = \overline{v}_0^N$, for $i = 1, \cdots, N$
- $\sup_{t\geq 0} \max_{i,j=1,\cdots,N} |x_t^i - x_t^j| < \infty$

Many extensions/refinements since original C-S contribution

A MFG FORMULATION

(**Nourian-Caines-Malhamé**)

$X_t^i = [x_t^i, v_t^i]$ state of player i

$$\begin{cases} dx_t^i & = v_t^i dt \\ dv_t^i & = [Av_t^i + B\alpha_t^i]dt + \sigma dW_t^i \end{cases}$$

For strategy profile $\underline{\alpha} = (\underline{\alpha}^1, \cdots, \alpha^N)$, the cost to player i

$$J^i(\alpha) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\frac{1}{2}|\alpha_t^i|^2 + \frac{1}{2}\left| \frac{1}{N} \sum_{j=1}^N w(|x_t^i - x_t^j|)[v_t^i - v_t^j] \right|^2 \right) dt$$

- **Ergodic** (infinite horizon) model;
- $\beta = 0$, **Linear Quadratic** (LQ) model;
- if $\beta > 0$, **asymptotic expansions** for $\beta << 1$?

REFORMULATION

$$J^i(\alpha) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f^i(t, X_t, \overline{\mu}_t^N, \alpha_t)dt$$

with

$$f^i(t, X, \mu, \underline{\alpha}) = \frac{1}{2}|\alpha^i|^2 + \frac{1}{2}\left|\int w(|x - x'|)[v - v']\mu(dx')\right|^2$$

where $X = [x, v]$ and $X' = [x, v]$.

Unfortunately

f^i is not convex !

More Examples of Interactions

- **Rank Effects**
  - $f(t, x, \mu, q, a)$ contains $G(\mu_t(-\infty, x_t])$
  - Oil production model (Guéant-Lasry-Lions)
- **Quantile Interactions**
  - $f(t, x, \mu, q, a)$ involves the quantile function $y \hookrightarrow F_{\mu_t}^{-1}(y) = \inf\{x \in \mathbb{R};\ \mu_t(-\infty, x] \geq y\}$
- **Functions of the *Density* of the Population** à la Lasry - Lions

Mean Field Games in Random Environment

Mean zero **Gaussian measure** $\underline{W} = (W(A,B))_{A\subset\Xi, B\subset[0,\infty)}$

$$\mathbb{E}[W(A,B)W(A',B')] = \nu(A\cap A')|B\cap B'|$$

where

- $|B|$ is Lebesgue measure of B
- ν is a non-negative measure on Ξ (**intensity**)

$$dX_t^i = b(t,X_t^i,\overline{\mu}_t^N,\alpha_t^i)dt + \sigma(t,X_t^i,\overline{\mu}_t^N,\alpha_t^i)dW_t^i + \int_\Xi c(t,X_t^i,\overline{\mu}_t^N,\alpha_t^i,\xi)W(d\xi,dt)$$

for $c : [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A^i\times\mathbb{R}^d \hookrightarrow \mathbb{R}^d$.

- If $c(x,\mu,\alpha,\xi) \sim c(t,x,\mu)\delta(x-\xi)$

$$\int_{\mathbb{R}^d} c(X_t^i,\overline{\mu}_t^N,\alpha_t^i,\xi)W(d\xi,dt) = c(t,X_t^i,\overline{\mu}_t^N)W(X_t^i,dt)$$

  (realistic in the case of the Cucker-Smale **flocking model**)
- If c independent of ξ and $W(d\xi,dt) = W(dt)$ (***common noise***)

Games with Major and Minor Players

More sophisticated model for banking network

$$\begin{cases} dX_t^{0,N} &= b^0(t, X_t^{0,N}, \overline{\mu}_t^N, \alpha_t^{0,N})dt + \sigma^0 dW_t^0 \\ dX_t^{i,N} &= b(t, X_t^{i,N}, \overline{\mu}_t^N, X_t^{0,N}, \alpha_t^{i,N})dt + \sigma dW_t^i, \qquad i = 1, 2, \cdots, N. \end{cases}$$

with cost functions

$$\begin{cases} J^{0,N}(\underline{\alpha}) = \mathbb{E}\Big[\int_0^T f^0(t, X_t^{0,N}, \overline{\mu}_t^N, \alpha_t^{0,N})dt + g^0(X_T^{0,N}, \overline{\mu}_T^N) \Big] \\ J^{i,N}(\underline{\alpha}) = \mathbb{E}\Big[\int_0^T f(t, X_t^{i,N}, \overline{\mu}_t^N, X_t^{0,N}, \alpha_t^{i,N})dt + g^0(X_T^{i,N}, \overline{\mu}_T^N, X_T^{0,N}) \Big] \end{cases}$$

- First take for minor players: Mean Field Game **conditioned** by major player
- Introduced by M. Huang for a particular LQ model

The Mean Field Game Strategy and the MFG Problem

OPTIMIZATION PROBLEM

Simultaneous Minimization of

$$J^i(\underline{\alpha}) = \mathbb{E}\left\{\int_0^T f(t, X_t^i, \overline{\mu}_t^N, \alpha_t^i)dt + g(X_T, \overline{\mu}_T^N)\right\}, \quad i = 1, \cdots, N$$

under **constraints** of the form

$$dX_t^i = b(t, X_t^i, \overline{\mu}_t^N, \alpha_t^i)dt + \sigma dW_t^i, \quad i = 1, \cdots, N.$$

GOAL: search for **equilibriums**

Model Requirements

- Each player **cannot** on its own, influence **significantly** the global output of the game
- **Large** number of **statistically identical** players ($N \to \infty$)
- **Closed loop** controls in **feedback** form

$$\alpha_t^i = \phi^i(t, (X_t^1, \cdots, X_t^N)), \qquad i = 1, \cdots, N.$$

- **Distributed** controls

$$\alpha_t^i = \phi^i(t, X_t^i), \qquad i = 1, \cdots, N.$$

- **Identical** feedback functions

$$\phi^1(t, \cdot\,) = \cdots = \phi^N(t, \cdot\,) = \phi(t, \cdot\,), \qquad 0 \le t \le T.$$

TOUTED SOLUTION (WISHFUL THINKING)

- **Identify** a (distributed closed loop) **strategy** ϕ from **effective equations** (from stochastic optimization for large populations)
- Each player is assigned the same function ϕ
- At each time t, player i take action $\alpha_i = \phi(t, X_t^i)$

What is the resulting **population behavior**?

- Did we reach some form of equilibrium?
- If yes, what kind of equilibrium?

Mean Field Game (MFG) Strategy

- By **symmetry**, interactions depend upon

  empirical distributions
- When constructing the **best response map**

  ALL stochastic optimizations should be "**the same**"
- When N is **large**
  - empirical distributions should converge
  - capture interactions with limits of empirical distributions
  - **ONE** standard stochastic control problem **for each possible limit**
- Still need a **fixed point** for choice of the **limit distribution** to be the right one

Lasry - Lions (MFG) **Caines - Malhamé - Huang** (NCE)

Summary of the MFG Approach

1. Fix a deterministic function $[0,T] \ni t \hookrightarrow \mu_t \in \mathcal{P}(\mathbb{R})$;
2. Solve the standard stochastic control problem

$$\phi^* = \arg\inf_{\phi} \mathbb{E}\left\{ \int_0^T f(t, X_t, \mu_t, \phi(t, X_t))dt + g(X_T, \mu_T) \right\}$$

   subject to

$$dX_t = b(t, X_t, \mu_t, \phi(t, X_t))dt + \sigma dW_t;$$

3. Determine the function $[0,T] \ni t \hookrightarrow \mu_t \in \mathcal{P}(\mathbb{R})$ so that

$$\forall t \in [0,T], \quad \mathbb{P}_{X_t} = \mu_t.$$

Once this is done,

$$\alpha_t^{j*} = \phi^*(t, X_t^j), \qquad j = 1, \cdots, N$$

form an **approximate Nash equilibrium** for the game.

MFG Adjoint Equations

Mean Field Interaction thru the states X_t^i ONLY
σ constant for simplicity

Freeze $\mu = (\mu_t)_{0\le t\le T}$, write (reduced) Hamiltonian

$$H^{\mu_t}(t,x,y,\alpha) = b(t,x,\mu_t,\alpha)\cdot y + f(t,x,\mu_t,\alpha)$$

Given an admissible control $\underline{\alpha} = (\alpha_t)_{0\le t\le T}$ and the corresponding controlled state process $X^\alpha = (X_t^\alpha)_{0\le t\le T}$, any couple $(Y_t, Z_t)_{0\le t\le T}$ satisfying:

$$\begin{cases} dY_t = -\partial_x H^{\mu_t}(t, X_t^\alpha, Y_t, \alpha_t)dt + Z_t dW_t \\ Y_T = \partial_x g(X_T^\alpha, \mu_T) \end{cases}$$

is called a set of **adjoint processes**

Stochastic Maximum Principle (Pontryagin)

Determine

$$\hat{\alpha}^{\mu_t}(t,x,y) = \arg\inf_{\alpha\in A} H^{\mu_t}(t,x,y,\alpha)$$

Inject in **FORWARD** and **BACKWARD** dynamics and **SOLVE**

$$\begin{cases} dX_t = b(t, X_t, \mu_t, \hat{\alpha}^{\mu}(t, X_t, Y_t))dt + \sigma dW_t, \quad X_0 = x_0 \\ dY_t = -\partial_x H^{\mu_t}(t, X_t, Y_t, \hat{\alpha}^{\mu_t}(t, X_t, Y_t))dt + Z_t dW_t, \quad Y_T = \partial_x g(X_T, \mu_t) \end{cases}$$

Standard **FBSDE** (for each fixed $t \hookrightarrow \mu_t$)

Fixed Point Step

Solve the **fixed point problem**

$$(\mu_t)_{0\le t\le T} \quad\longrightarrow\quad (X_t)_{0\le t\le T} \quad\longrightarrow\quad (\mathbb{P}_{X_t})_{0\le t\le T}$$

Note: if we enforce $\mu_t = \mathbb{P}_{X_t}$ for all $0 \le t \le T$ in FBSDE we have

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t}, \hat{\alpha}^{\mathbb{P}_{X_t}}(t, X_t, Y_t))dt + \sigma dW_t, \quad X_0 = x_0 \\ dY_t = -\partial_x H^{\mathbb{P}_{X_t}}(t, X_t^{\alpha}, Y_t, \hat{\alpha}^{\mathbb{P}_{X_t}}(t, X_t, Y_t))dt + Z_t dW_t, \quad Y_T = \partial_x g(X_T, \mathbb{P}_{X_T}) \end{cases}$$

FBSDE of McKean-Vlasov type !!!

Aside: Solution of McKean-Vlasov FBSDEs

Existence of a solution of

$$\begin{cases} dX_t = b(t, X_t, Y_t, \mathbb{P}_{(X_t,Y_t)})dt + \sigma(t, X_t, Y_t, \mathbb{P}_{(X_t,Y_t)})dW_t \\ dY_t = -\Psi(t, X_t, Y_t, \mathbb{P}_{(X_t,Y_t)})dt + Z_t dW_t \\ X_0 = x, \; Y_T = g(X_T, \mathbb{P}_{X_T}) \end{cases}$$

if coefficients are **uniformly Lipschitz** and **bounded**

boundedness assumption **can be** relaxed

e.g. MFG and Controlled McKean-Vlasov models (later on in the lectures)

Proof works for $\mathbb{P}_{(X_t,Y_t,Z_t)}$ instead of $\mathbb{P}_{(X_t,Y_t)}$

Solution of the MFG Problem

Assumptions

- Convex costs (f and g)
- Uncontrolled volatility ($\sigma(t, x, \mu, \alpha) \equiv \sigma > 0$)
- $b(t, x, \mu, \alpha) = b_0(t, \mu) + b_1(t)x + b_2(t)\alpha$ with bounded b_i's

Then

$$\hat{\alpha}(t, x, y, \mu) \in \arg\inf_{\alpha} H^{\mu}(t, x, y, \alpha)$$

is Lip-1 in (x, y, μ) uniformly in $t \in [0, T]$ and one can solve:

$$\begin{cases} dX_t = b(t, X_t, Y_t, \mathbb{P}_{X_t}, \hat{\alpha}(t, X_t, Y_t, \mathbb{P}_{X_t}))dt + \sigma dW_t \\ dY_t = -\partial_x f(t, X_t, Y_t, \mathbb{P}_{X_t}, \hat{\alpha}(t, X_t, Y_t, \mathbb{P}_{X_t}))dt - b_1(t) Y_t + Z_t dW_t \\ X_0 = x, \; Y_T = \partial_x g(X_T, \mathbb{P}_{X_T}) \end{cases}$$

and the solution is of the form

$$Y_t = u(t, X_t)$$

Back to the N-Player (Mean Field) Game

:

$$dX_t^i = b(t, X_t^i, \overline{\mu}_t^N, \alpha_t^i)dt + \sigma dW_t^i, \qquad 0 \le t \le T, \quad 1 \le i \le N$$

where

$$\overline{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}.$$

Then the controls

$$\hat{\alpha}_t^i = \hat{\alpha}(t, X_t^i, \mathbb{P}_{X_t}, u(t, X_t^i))$$

form an **ϵ_N-Nash equilibrium** with $\epsilon_N \searrow 0$, as for each $1 \le i \le N$

$$J(\hat{\alpha}_t^1, \cdots, \alpha_t^i, \cdots, \hat{\alpha}_t^N) \ge J(\hat{\alpha}_t^1, \cdots, \hat{\alpha}_t^i, \cdots, \hat{\alpha}_t^N) - \epsilon_N$$

The Weak Formulation for Mean Field Games

First Set of Assumptions

- The *control space* A is a compact convex;
- All progressively measurable A-valued processes are admissible;
- Drift $b : [0, T] \times \mathcal{C} \times \mathbb{P}_\psi(\mathcal{C}) \times A \to \mathbb{R}^d$ progressively measurable, continuous in μ.
- Volatility $\sigma : [0, T] \times \mathcal{C} \to \mathbb{R}^{d \times d}$ progressively measurable.
- There exists a unique strong solution X of the driftless state equation

$$dX_t = \sigma(t, X) dW_t, \qquad X_0 = \xi$$

  such that $\mathbb{E}[\psi^2(X)] < \infty$,
- $\sigma(t, X) > 0$ for all $t \in [0, T]$ almost surely,
- $\sigma^{-1}(t, X) b(t, X, \mu, a)$ is bounded.

Weak Formulation

For each $\mu \in \mathbb{P}_\psi(\mathcal{C})$ and admissible $\alpha \in \mathbb{A}$, define

- $\diamond$ the probability $\mathbb{P}^{\mu,\alpha}$ on $(\Omega, \mathcal{F}_T)$ by

$$\frac{d\mathbb{P}^{\mu,\alpha}}{d\mathbb{P}} = \exp\left[\int_0^T \sigma^{-1} b(t, X, \mu, \alpha_t) \cdot dW_t - \frac{1}{2}\int_0^T \left|\sigma^{-1} b(t, X, \mu, \alpha_t)\right|^2 dt\right].$$

- $\diamond$ the process $W^{\mu,\alpha}$ defined by

$$W_t^{\mu,\alpha} := W_t - \int_0^t \sigma^{-1} b(s, X, \mu, \alpha_s)\, ds$$

- $\diamond$ so that

$$dX_t = b(t, X, \mu, \alpha_t)\, dt + \sigma(t, X) dW_t^{\mu,\alpha}.$$

Weak Formulation (cont.)

- Running objective $f : [0,T] \times \mathcal{C} \times \mathcal{P}_\psi(\mathcal{C}) \times \mathcal{P}(A) \times A \to \mathbb{R}$ of the form
$$f(t,x,\mu,q,a) = f_1(t,x,\mu,a) + f_2(t,x,\mu,q).$$
- Terminal objective $g : \mathcal{C} \times \mathcal{P}_\psi(\mathcal{C}) \to \mathbb{R}$ is measurable

$$|g(x,\mu)|+|f(t,x,\mu,q,a)| \le c\left(\psi(x) + \rho\left(\int \psi \, d\mu\right)\right), \quad \forall (t,x,\mu,q,a).$$

for $c > 0$ and an increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$

Problem Statement

Given

- a measure $\mu \in \mathcal{P}(\mathcal{C})$
- a measurable map $[0, T] \ni t \mapsto q_t \in \mathcal{P}(A)$

define the associated conditional expected reward for $\alpha \in \mathbb{A}$ by

$$J_t^{\mu,q}(\alpha) := \mathbb{E}^{\mu,\alpha}\left[\int_t^T f(s, X, \mu, q_s, \alpha_s)ds + g(X, \mu)\middle| \mathcal{F}_t\right]$$

and the conditional value function by

$$V_t^{\mu,q} = \inf_{\alpha \in \mathbb{A}} J_t^{\mu,q}(\alpha).$$

Goal: **Find μ and q s.t.**

- **there exists $\hat{\alpha} \in \mathbb{A}$ such that $V_0^{\mu,q} = J_0^{\mu,q}(\hat{\alpha})$,**
- **$\mathbb{P}^{\mu,\hat{\alpha}} \circ X^{-1} = \mu$, and $\mathbb{P}^{\mu,\hat{\alpha}} \circ \hat{\alpha}_t^{-1} = q_t$ for almost every t**

Existence and uniqueness

Hamiltonian $h : [0, T] \times \mathcal{C} \times \mathcal{P}_\psi(\mathcal{C}) \times \mathcal{P}(A) \times \mathbb{R}^d \times A \to \mathbb{R}$,

$$h(t, x, \mu, q, z, a) = f(t, x, \mu, q, a) + z \cdot \sigma^{-1} b(t, x, \mu, a)$$

Maximized Hamiltonian $H : [0, T] \times \mathcal{C} \times \mathcal{P}_\psi(\mathcal{C}) \times \mathcal{P}(A) \times \mathbb{R}^d \to \mathbb{R}$

$$H(t, x, \mu, q, z) := \sup_{a \in A} h(t, x, \mu, q, z, a)$$

Arg-max set

$$A(t, x, \mu, z) := \{a \in A : h(t, x, \mu, q, z, a) = H(t, x, \mu, q, z)\}$$

- $A(t, x, \mu, z)$ does not depend upon q
- $A(t, x, \mu, z)$ is not empty

Finally, a BSDE !

$$Y_t^{\mu,\nu} = g(X,\mu) + \int_t^T H(s, X, \mu, \nu_s, Z_s^{\mu,\nu})ds - \int_t^T Z_s^{\mu,\nu} \cdot dW_s$$

For each $\alpha \in \mathbb{A}$, we may also solve the BSDE

$$Y_t^{\mu,\nu,\alpha} = g(X,\mu) + \int_t^T h(s, X, \mu, \nu_s, Z_s^{\mu,\nu,\alpha}, \alpha_s)ds - \int_t^T Z_s^{\mu,\nu,\alpha} \cdot dW_s$$

$$= g(X,\mu) + \int_t^T f(s, X, \mu, \nu_s, \alpha_s)ds - \int_t^T Z_s^{\mu,\nu,\alpha} \cdot dW_s^{\mu,\alpha}.$$

and since $W^{\mu,\alpha}$ is a Wiener process under $\mathbb{P}^{\mu,\alpha}$ and $Y^{\mu,\alpha}$ is adapted

$$Y_t^{\mu,\nu,\alpha} = \mathbb{E}^{\mu,\alpha}\left[g(X,\mu) + \int_t^T f(s, X, \mu, \nu, \alpha_s)ds \,\Big|\, \mathcal{F}_t^n \right] = J_t^{\mu,\nu}(\alpha).$$

- By comparison principle $Y_t^{\mu,\nu} \geq V_t^{\mu,\nu}$
- By measurable selection, there exists $\hat{\alpha} : [0,T] \times \mathcal{C} \times \mathcal{P}_\psi(\mathcal{C}) \times \mathcal{P}(A) \times \mathbb{R}^d \to A$

$$H(t, x, \mu, \nu, z) = h(s, x, \mu, \nu, z, \hat{\alpha}(t, x, \mu, z)), \quad \text{for all } (t, x, \mu, \nu, z),$$

The process $\alpha^{\mu,\nu}$

$$\alpha_t^{\mu,\nu} := \hat{\alpha}(t, X, \mu, Z_t^{\mu,\nu})$$

is an optimal control, **but so is any process in the set**

$$\mathcal{A}(\mu,\nu) := \{\alpha \in \mathbb{A} : H(t, X, \mu, \nu_t, y) = h(t, X, \mu, \nu_t, Z_t^{\mu,\nu}, \alpha_t) \ dt \times d\mathbb{P} - a.e.\}$$

Final Step

Define $\Phi : \mathcal{P}_\psi(\mathcal{C}) \times \mathbb{A} \to \mathcal{P}(\mathcal{C}) \times \mathcal{M}$ by

$$\Phi(\mu, \alpha) := (\mathbb{P}^{\mu,\alpha} \circ X^{-1}, \delta_{\mathbb{P}^{\mu,\alpha} \circ \alpha_t^{-1}}(dq)dt)$$

The goal now is to find a point $(\mu, \nu) \in \mathcal{P}_\psi(\mathcal{C}) \times \mathcal{M}$ for which there exists $\alpha \in \mathcal{A}(\mu, \nu)$ such that $(\mu, \nu) = \Phi(\mu, \alpha)$. In other words, we seek a fixed point of the set-valued map $(\mu, \nu) \mapsto \Phi(\mu, \mathcal{A}(\mu, \nu)) := \{\Phi(\mu, \alpha) : \alpha \in \mathcal{A}(\mu, \nu)\}$.

McKean-Vlasov FBSDEs: Wishful Thinking !

Main difficulty is the analysis is the adjoint process $Z^{\mu,\nu}$.

For each (μ, ν), $Z_t^{\mu,\nu} = \zeta_{\mu,\nu}(t, X)$ and if $\hat{\alpha}$ is a measurable selection as before, **any solution of**

$$\begin{cases} dX_t = b(t, X, \mu, \hat{\alpha}(t, X, \mu, \zeta_{\mu,\nu}(t, X)))dt + \sigma(t, X)dW_t, \\ X \quad \sim \mu, \ \mu \circ (\hat{\alpha}(t, \cdot, \mu, \zeta_{\mu,\nu}(t, \cdot)))^{-1} = \nu_t \ a.e. \end{cases}$$

is a solution of our MFG problem

Can't solve this McKean-Vlasov SDE!

Some (loosely stated) Results

Theorem

- *If b, f, g are continuous in (μ, ν, α), the Hamiltonian h is concave in α, some growth conditions hold and $f = f_1(t, x, \mu, a) + f_2(t, x, \mu, \nu)$, then **there exists a fixed point**.*
- *if the Hamiltonian h is strictly concave in α, $f = f_1(t, \mu, \nu) + f_2(t, x, a)$, and $b = b(t, x, a)$, then **the fixed point is unique**.*

Approximate equilibria for the finite-player game

Theorem

*If $\alpha = \alpha(t, X_\cdot)$ is an optimal feedback control for the MFG problem, then the strategy profiles $\alpha(t, X_\cdot^i)$ form an **approximate Nash equilibrium** for the finite-player game (i.e. for some $\epsilon_n \downarrow 0$, no player can increase his expected reward by more than ϵ_n by unilaterally changing strategy).*

Price Impact Model Revisited

Price impact model corresponds to

- $b(t, x, \mu, \alpha) = \alpha$;
- σ constant;
- $g(x, \mu) = G(x)$;
- $f(t, x, \mu, \nu, \alpha) = \gamma x \int c' d\nu - c(\alpha) - F(t, x)$.

Theorem

For a bounded order book, with c' continuous, the mean field price impact model has a solution. Moreover, the errors ϵ_n are $O(1/\sqrt{n})$.

Control of McKean - Vlasov Dynamics

Franchise Equilibrium

We say that $(t, x) \hookrightarrow \phi^*(t, x)$ gives a **franchise equilibrium** if

$$\phi^* = \arg\inf_{\phi} \mathbb{E}\left\{ \int_0^T f(t, X_t^i, \overline{\mu}_t^N, \phi(t, X_t^i))dt + g(X_T, \overline{\mu}_T^N) \right\}.$$

where for each player $i \in \{1, \cdots, N\}$ we have $\alpha_t^i = \phi(t, X_t^i)$.

So when one player perturbs his/her ϕ

ALL players perturb their ϕ's in the same way!

So the streamlining procedure is

1. Take the limit $N \to \infty$ (i.e. solve the **fixed point problem**) **FIRST**
2. Solve the optimization problem **NEXT**

TAKING THE LIMIT $N \to \infty$ FIRST

Propagation of Chaos
(**Mc Kean / Sznitmann / Jourdain-Méleard-Woyczinski**)

- Focus on N_0 (fixed) player in a large set ($N \to \infty$) of players
- Their private state processes X_t^j for $j = 1, \cdots, N_0$ become
  - (Asymptotically) **independent identically distributed**
  - (Asymptotically) **distributed** like the solution of (McKV)

$$dX_t = b(t, X_t, \mathbb{P}_{X_t}, \phi(t, X_t))dt + \sigma d\tilde{W}_t$$

The individual objective costs become

$$J(\phi) = \mathbb{E}\left\{\int_0^T f(t, X_t, \mathbb{P}_{X_t}, \phi(t, X_t))dt + g(X_T, \mathbb{P}_{X_T})\right\}$$

Control of McKean-Vlasov Dynamics

Stochastic optimization problem: minimize

$$J(\underline{\alpha}) = \mathbb{E}\left[\int_0^T f(t, X_t, \mathbb{P}_{X_t}, \alpha_t)dt + g(X_T, \mathbb{P}_{X_T})\right],$$

over admissible control processes $\underline{\alpha} = (\alpha_t)_{0 \leq t \leq T}$ subject to

$$dX_t = b(t, X_t, \mathbb{P}_{X_t}, \alpha_t)dt + \sigma(t, X_t, \mathbb{P}_{X_t}, \alpha_t)dW_t \qquad 0 \leq t \leq T,$$

- **PDE approach** difficult
  - X_t not Markovian
  - $(X_t, \mathbb{P}_{X_t})$ evolves in an infinite dimensional manifold
- **Probabilistic approach** (**stochastic maximum principle**) **Hamiltonian**

$$H(t, x, \mu, y, z, \alpha) = b(t, x, \mu, \alpha) \cdot y + \sigma(t, x, \mu, \alpha) \cdot z + f(t, x, \mu, \alpha)$$

(Informal) Natural Question

Is the diagram

$$
\begin{array}{ccc}
\begin{array}{c}\text{SDE State Dynamics}\\ \text{for N players}\end{array} & \underset{\text{Optimization}}{\longrightarrow} & \begin{array}{c}\text{Nash Equilibrium}\\ \text{for N players}\end{array} \\
\downarrow \begin{array}{c}(\textit{fixed point})\\ N\to\infty\end{array} & & \downarrow \begin{array}{c}(\textit{fixed point})\\ N\to\infty\end{array} \\
\text{McKean Vlasov Dynamics} & \overset{\text{Optimization}}{\longrightarrow} & \begin{array}{c}\text{Mean Field Game?}\\ \text{Controlled McK-V Dynamics?}\end{array}
\end{array}
$$

commutative?

Differentiability and Convexity of $\mu \hookrightarrow h(\mu)$

- Notions of differentiability for functions defined on spaces of measures from theory of optimal transportation, gradient flows, etc) studied by **Ambrosio, De Giorgi, Otto, Villani**, etc
- Tailored made notion (**Lions**' Collège de France Lectures, **Cardaliaguet**)

Lift a function $\mu \hookrightarrow h(\mu)$ to $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ into

$$X \hookrightarrow \tilde{h}(X) = h(\tilde{\mathbb{P}}_X)$$

and say

h is differentiable at μ if $\tilde{h}$ is Fréchet differentiable at X whenever $\tilde{\mathbb{P}}_X = \mu$.

A function g on $\mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$ is said to be **convex** if for every (x, μ) and (x', μ') in $\mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$ we have

$$g(x', \mu') - g(x, \mu) - \partial_x g(x, \mu) \cdot (x' - x) - \tilde{\mathbb{E}}[\partial_\mu g(x, \tilde{X}) \cdot (\tilde{X}' - \tilde{X})] \geq 0$$

whenever $\tilde{\mathbb{P}}_{\tilde{X}} = \mu$ and $\tilde{\mathbb{P}}_{\tilde{X}'} = \mu'$

The Adjoint Equations

Lifted Hamiltonian

$$\tilde{H}(t, x, \tilde{X}, y, \alpha) = H(t, x, \mu, y, \alpha)$$

for any random variable $\tilde{X}$ with distribution μ.

Given an admissible control $\underline{\alpha} = (\alpha_t)_{0 \le t \le T}$ and the corresponding controlled state process $\underline{X}^\alpha = (X_t^\alpha)_{0 \le t \le T}$, any couple $(Y_t, Z_t)_{0 \le t \le T}$ satisfying:

$$\begin{cases} dY_t = -\partial_x H(t, X_t^\alpha, \mathbb{P}_{X_t^\alpha}, Y_t, \alpha_t)dt + Z_t dW_t \\ \qquad\qquad -\tilde{\mathbb{E}}[\partial_\mu \underline{H}(t, \tilde{X}_t, X, \tilde{Y}_t, \tilde{\alpha}_t)]|_{X = X_t^\alpha} dt \\ Y_T = \partial_x g(X_T^\alpha, \mathbb{P}_{X_T^\alpha}) + \tilde{\mathbb{E}}[\partial_\mu g(x, \tilde{X}_t)]|_{x = X_T^\alpha} \end{cases}$$

where $(\tilde{\alpha}, \tilde{X}, \tilde{Y}, \tilde{Z})$ is an independent copy of (α, X^α, Y, Z), is called a set of **adjoint processes**

BSDE of Mean Field type according to Buckhdan-Li-Peng !!!

Extra terms in red are the ONLY difference between MFG and Control of McKean-Vlasov dynamics !!!

A Necessary Condition for Optimality

If $\underline{X} = \underline{X}^{\underline{\alpha}}$ controlled McKean-Vlasov dynamics ($X_0 = x$), compute the **Gâteaux derivative of the cost functional** J at $\underline{\alpha}$ in the direction of $\underline{\beta}$ using dual processes and the variation process $\underline{V} = (V_t)_{0 \le t \le T}$ solution of the equation

$$dV_t = [\gamma_t V_t + \delta_t(\mathbb{P}_{(X_t,V_t)}) + \eta_t]dt + [\tilde{\gamma}_t V_t + \tilde{\delta}_t(\mathbb{P}_{(X_t,V_t)}) + \tilde{\eta}_t]dW_t$$

where the coefficients γ_t, δ_t, η_t, $\tilde{\gamma}_t$, $\tilde{\delta}_t$ and $\tilde{\eta}_t$ are defined as

$$\begin{aligned} \gamma_t &= \partial_x b(t, X_t, \mathbb{P}_{X_t}, \alpha_t), \qquad \text{and} \qquad \tilde{\gamma}_t = \partial_x \sigma(t, X_t, \mathbb{P}_{X_t}, \alpha_t) \\ \eta_t &= \partial \alpha b(t, X_t, \mathbb{P}_{X_t}, \alpha_t)\beta_t, \qquad \text{and} \qquad \tilde{\eta}_t = \partial_\alpha \sigma(t, X_t, \mathbb{P}_{X_t}, \alpha_t)\beta_t \\ \gamma_t &= \partial_x b(t, X_t, \mathbb{P}_{X_t}, \alpha_t), \qquad \text{and} \qquad \tilde{\gamma}_t = \partial_x \sigma(t, X_t, \mathbb{P}_{X_t}, \alpha_t) \end{aligned}$$

and

$$\delta_t = \tilde{\mathbb{E}} \partial_\mu b(t, x, \mathbb{P}_{X_t}, \alpha)(\tilde{X}_t) \cdot \tilde{V}_t \Big|_{\substack{x = X_t \\ \alpha = \alpha_t}}, \qquad \text{and} \qquad \tilde{\delta}_t = \tilde{\mathbb{E}} \partial_\mu \sigma(t, x, \mathbb{P}_{X_t}, \alpha)(\tilde{X}_t) \cdot \tilde{V}_t \Big|_{\substack{x = X_t \\ \alpha = \alpha_t}}$$

where $(\tilde{X}_t, \tilde{V}_t)$ is an independent copy of (X_t, V_t).

Pontryagin Minimum Principle (Sufficiency)

Assume

1. Coefficients continuously differentiable with bounded derivatives;
2. Terminal cost function g is convex;
3. α admissible control, X corresponding dynamics, (Y, Z) adjoint processes and

$$(x, \mu, \alpha) \hookrightarrow H(t, x, \mu, Y_t, Z_t, \alpha)$$

   is $dt \otimes d\mathbb{P}$ a.e. **convex**,

then, if moreover

$$H(t, X_t, \mathbb{P}_{X_t}, Y_t, Z_t, \alpha_t) = \inf_{\alpha \in A} H(t, X_t, \mathbb{P}_{X_t}, Y_t, \alpha), \quad \text{a.s.}$$

Then α is an optimal control, i.e.

$$J(\alpha) = \inf_{\overline{\alpha} \in \mathcal{A}} J(\overline{\alpha})$$

Scalar Interactions

$$b(t,x,\mu,\alpha)=\tilde b(t,x,\langle\psi,\mu\rangle,\alpha) \quad \sigma(t,x,\mu,\alpha)=\tilde\sigma(t,x,\langle\phi,\mu\rangle,\alpha)$$
$$f(t,x,\mu,\alpha)=\tilde f(t,x,\langle\gamma,\mu\rangle,\alpha) \quad g(x,\mu)=\tilde g(x,\langle\zeta,\mu\rangle)$$

- ψ, ϕ, γ and ζ **differentiable** with at most quadratic growth at ∞,
- $\tilde b$, $\tilde\sigma$ and $\tilde f$ **differentiable** in $(x,r)\in\mathbb{R}^d\times\mathbb{R}$ for $t,\alpha)$ fixed
- $\tilde g$ **differentiable** in $(x,r)\in\mathbb{R}^d\times\mathbb{R}$.

Recall that the adjoint process satisfies

$$Y_T=\partial_x g(X_T,\mathbb{P}_{X_T})+\tilde{\mathbb{E}}[\partial_\mu g(\tilde X_T,\mathbb{P}_{\tilde X_T})(X_T)].$$

but since

$$\partial_\mu g(x,\mu)(x')=\partial_r\tilde g\big(x,\langle\zeta,\mu\rangle\big)\partial\zeta(x'),$$

the terminal condition reads

$$Y_T=\partial_x\tilde g\big(X_T,\mathbb{E}[\zeta(X_T)]\big)+\tilde{\mathbb{E}}\Big[\partial_r\tilde g\big(\tilde X_T,\mathbb{E}[\zeta(X_T)]\big)\Big]\partial\zeta(X_T)$$

Convexity in μ follows convexity of $\tilde g$

Scalar Interactions (cont.)

$$H(t,x,\mu,y,z,\alpha) = \tilde b(t,x,\langle\psi,\mu\rangle,\alpha)\cdot y + \tilde\sigma(t,x,\langle\phi,\mu\rangle,\alpha)\cdot z + \tilde f(t,x,\langle\gamma,\mu\rangle,\alpha).$$

$\partial_\mu H(t,x,\mu,y,z,\alpha)$ can be identified wih

$$\begin{aligned}\partial_\mu H(t,x,\mu,y,z,\alpha)(x') = & \big[\partial_r \tilde b(t,x,\langle\psi,\mu\rangle,\alpha)\cdot y\big]\partial\psi(x') \\ & + \big[\partial_r\tilde\sigma(t,x,\langle\phi,\mu\rangle,\alpha)\cdot z\big]\partial\phi(x') \\ & + \partial_r \tilde f(t,x,\langle\gamma,\mu\rangle,\alpha)\,\partial\gamma(x')\end{aligned}$$

and the adjoint equation rewrites:

$$\begin{aligned}dY_t = -\Big\{ & \partial_x \tilde b(t,X_t,\mathbb{E}[\psi(X_t)],\alpha_t)\cdot Y_t + \partial_x\tilde\sigma(t,X_t,\mathbb{E}[\phi(X_t)],\alpha_t)\cdot Z_t \\ & + \partial_x \tilde f(t,X_t,\mathbb{E}[\gamma(X_t)],\alpha_t)\Big\}dt + Z_t dW_t \\ -\Big\{ & \tilde{\mathbb{E}}\big[\partial_r\tilde b(t,\tilde X_t,\mathbb{E}[\psi(\tilde X_t)],\tilde\alpha_t)\cdot\tilde Y_t\big]\partial\psi(X_t) + \tilde{\mathbb{E}}\big[\partial_r\tilde\sigma(t,\tilde X_t,\mathbb{E}[\phi(\tilde X_t)],\tilde\alpha_t)\cdot\tilde Z_t\big]\partial\phi(X_t) \\ & + \tilde{\mathbb{E}}\big[\partial_r\tilde f((t,\tilde X_t,\mathbb{E}[\gamma(\tilde X_t)],\tilde\alpha_t)\big]\partial\gamma(X_t)\Big\}dt\end{aligned}$$

Anderson - Djehiche

Solution of the McKV Control Problem

Assume

- $b(t,x,\mu,\alpha) = b_0(t)\int_{\mathbb{R}^d} x d\mu(x) + b_1(t)x + b_2(t)\alpha$
  with b_0, b_1 and b_2 is $\mathbb{R}^{d\times d}$-valued and are bounded.
- f and g as in MFG problem.

There exists a solution $(X_t, Y_t, Z_t)_0$ of the McKean-Vlasov FBSDE

$$\begin{cases} dX_t = b_0(t)\mathbb{E}(X_t)dt + b_1(t)X_t dt + b_2(t)\hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)dt + \sigma dW_t, \\ dY_t = -\partial_x H\big(t, X_t, \mathbb{P}_{X_t}, Y_t, \hat{\alpha}_t\big)dt \\ \qquad\qquad - \mathbb{E}\big[\partial_\mu \underline{H}(t, X_t', X_t, Y_t', \hat{\alpha}_t')\big]dt + Z_t dW_t. \end{cases}$$

with $Y_t = u(t, X_t, \mathbb{P}_{X_t})$ for a function

$$u : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \ni (t,x,\mu) \mapsto u(t,x,\mu)$$

uniformly of Lip-1 and with linear growth in x.

A Finite Player Approximate Equilibrium

For N independent Brownian motions $(W^1, \dots, W^N)$ and for a square integrable exchangeable process $\beta = (\beta^1, \dots, \beta^N)$, consider the system

$$dX_t^i = \frac{1}{N} b_0(t) \sum_{j=1}^{N} X_t^j + b_1(t) X_t^i + b_2(t)\beta_t^i + \sigma dW_t^i, \quad X_0^i = \xi_0^i,$$

and define the common cost

$$J^N(\beta) = \mathbb{E}\Big[\int_0^T f(s, X_s^i, \bar\mu_s^N, \beta_s^i) ds + g(X_T^1, \bar\mu_T^N)\Big], \quad \text{with } \bar\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}.$$

Then, there exists a sequence $(\epsilon_N)_{N \ge 1}$, $\epsilon_N \searrow 0$, s.t. **for all** $\beta = (\beta^1, \dots, \beta^N)$,

$$J^N(\beta) \ge J^N(\alpha) - \epsilon_N,$$

where, $\alpha = (\alpha^1, \cdots, \alpha^N)$ with

$$\alpha_t^i = \hat\alpha(s, \tilde X_t^i, u(t, \tilde X_t^i), \mathbb{P}_{X_t})$$

where X and u are from the solution to the **controlled McKean Vlasov problem**, and $(\tilde X^1, \dots, \tilde X^N)$ is the state of the system controlled by α, i.e.

$$d\tilde X_t^i = \frac{1}{N}\sum_{j=1}^{N} b_0(t)\tilde X_t^j + b_1(t)\tilde X_t^i + b_2(t)\hat\alpha(s, \tilde X_s^i, u(s, \tilde X_s^i), \mathbb{P}_{X_s}) + \sigma dW_t^i, \quad \tilde X_0^i = \xi_0^i.$$