String duality and K3 surfaces from Seiberg-Witten curves

Workshop on modular forms around string theory
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(joint work with Dave Morrison and with Chuck Doran)

Overview

1. String duality: Heterotic theory/F-theory
2. ‘Quantum’ compactifications of the heterotic string
3. Examples of K3’s associated with Seiberg-Witten curves (period maps = generalized hypergeometric functions)

Heterotic/F-theory Duality

The heterotic string compactified on an $(n-1)$-dimensional elliptically fibered Calabi-Yau $\pi_H : \mathbf{Z} \to \mathbf{B}$ is equivalent to F-theory compactified on an n-dimensional K3-fibered Calabi-Yau $\pi_F : \mathbf{X} \to \mathbf{B}$, which is also elliptically fibered with a section.

Eight-dimensional compactifications: $n = 2$ and $\mathbf{B} = \mathrm{pt}$

- Heterotic CY: $\mathbf{Z} = E$ elliptic curve w/ principal G-bundle, $G \subset (E_8 \times E_8) \rtimes \mathbb{Z}_2$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$.
- F-theoretic CY: elliptic K3-surface $\mathbf{X} \to \mathbb{CP}^1$ w/ section,

  $$\bar{\mathbf{X}} : \; Y^2 = 4\,X^3 - g_2\,X - g_3\,, \qquad g_2 \in H^0(\mathcal{O}(8)),\; g_3 \in H^0(\mathcal{O}(12)).$$

- Moduli spaces for both types are given by the Narain space

  $$\mathfrak{M} = \mathrm{SO}(2,18;\mathbb{Z})\Big\backslash \mathrm{SO}(2,18)\Big/\Big(\mathrm{SO}(2) \times \mathrm{SO}(18)\Big)\,.$$

F-theoretic description of type IIB string backgrounds

- Complex scalar τ with $\mathrm{Im}\tau > 0$ is allowed to be multi-valued, and it is defined away from defects of codimension two.
- $\mathrm{SL}(2,\mathbb{Z})$ acts by standard fractional linear action on τ.
- To describe the effective field theory one needs:
  1) $\mathrm{SL}(2,\mathbb{Z})$-invariant function $j(\tau)$ – functional invariant,
  2) the precise $\mathrm{SL}(2,\mathbb{Z})$ action on τ – homological invariant.
- Description in Weierstrass model: $Y^2 = 4\,X^3 - g_2\,X - g_3$,
  coefficients: $g_2 = \sum_{j=0}^{8} a_j\, t^j, g_3 = \sum_{j=0}^{12} b_j\, t^j$, $[t:1] \in \mathbb{CP}^1$,
  number of moduli = 9 + 13 - 3 - 1 =18,
  discriminant: $\Delta = g_2^3 - 27\, g_3^2$,
  functional invariant: $j = g_2^3/\Delta$,
  homological invariant: from vanishing order of g_2, g_3, Δ.

F-theoretic description of type IIB string backgrounds

- Singular fiber where $\Delta = g_2^3 - 27\, g_3^2$ vanishes.
- Kodaira's classification of singular fibers:

	$\mathrm{ord}_D(g_2)$	$\mathrm{ord}_D(g_3)$	$\mathrm{ord}_D(\Delta)$	singularity	monodromy
$I_n,\ n \geq 1$	0	0	n	A_{n-1}	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
$I_n^*,\ n \geq 0$	2	3	$n+6$	D_{n+4}	$\begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}$
III^*	3	≥ 5	9	E_7	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
II^*	≥ 4	5	10	E_8	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$
...	...	...	...	...	...

- Correspondence: string perspective $\leftrightarrow$ monodromy + $j = \frac{g_2^3}{\Delta}$
  IIB + n D7-branes $\leftrightarrow$ Kodaira type I_n, $j = \infty$,
  IIB + n D7-branes on O7-plane $\leftrightarrow$ Kodaira type I_n^*, $j = \infty$,
  IIB + exotic 7-branes $\leftrightarrow$ Kodaira type $III^*, II^*, \ldots, |j| < \infty$.

Heterotic string backgrounds

- Closed string theory on T^2 has two basic moduli:
  1) complex structure parameter $\rho \in \mathbb{H}$,
  2) complexified Kähler modulus $\sigma = B + i\, V \in \mathbb{H}$.
- **Geometric** compactifications:
  ρ varies over base, undergoes monodromies in $\mathrm{SL}(2,\mathbb{Z})$,
  σ is constant up to shifts.
- **Quantum** compactifications: ρ and σ vary over base,
  $\sigma \to -1/\sigma$ possible, inherently quantum.
- Moduli of heterotic string compactified on T^2 near boundary:

$$\underbrace{\begin{pmatrix}\text{complex str.}\\ \text{param.}\end{pmatrix}}_{\rho \circlearrowright \mathrm{SL}(2,\mathbb{Z})} \times \underbrace{\begin{pmatrix}\text{Kähler}\\ \text{param.}\end{pmatrix}}_{\sigma \circlearrowright \mathrm{SL}(2,\mathbb{Z})} \times \underbrace{\Big(\text{Wilson lines}\Big)}_{z,\ldots}$$

Matching the moduli: F-theory $\leftrightarrow$ heterotic string

- Understand duality on low-dimensional subspaces of $\mathfrak{M}$

$$\Gamma \Big\backslash \mathrm{SO}(2,r) \Big/ \Big(\mathrm{SO}(2) \times \mathrm{SO}(r)\Big) \subset \mathfrak{M}\,.$$

$$\Rightarrow \quad \begin{cases} r = 2: & (\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})) \rtimes \mathbb{Z}_2 \backslash (\mathbb{H} \times \mathbb{H}) \\ r = 3: & \mathrm{Sp}(4,\mathbb{Z}) \backslash \mathbb{H}_2 \end{cases}$$

- On heterotic side: gauge group G of high rank, i.e., compactifications w/ very few Wilson lines.

$r = 2$: No Wilson lines. $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ or $G = \mathrm{Spin}(32)/\mathbb{Z}_2$.
$r = 3$: One Wilson line. $G = E_8 \times E_7$ or $G = \mathrm{Spin}(28) \times \mathrm{SU}(2)/\mathbb{Z}_2$.

- On F-theory side: families of Jacobian K3 surfaces represent r-dimensional moduli spaces of lattice-polarized K3 surfaces.

- Use Shioda-Inose correspondence as the duality map between polarized K3 surface and principally polarized Abelian surface.

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- On F-theory side: families of Jacobian K3 surfaces represent r-dimensional moduli spaces of lattice-polarized K3 surfaces

- Use Shioda-Inose correspondence as the duality map between F-theory and heterotic string vacua.

Weierstrass fibrations → lattice polarized K3's

(based on Morrison-Vafa '96; Clingher-Doran '07, '10; A. Kumar '08)

$$\bar{\mathbf{X}} \to \mathbb{CP}^1 : \quad \left(\begin{array}{ccl} Y^2 & = & 4\,X^3 + \left(a\,t^4 \right) X \\ & + & \left(t^7 + b\,t^6 + d\,t^5 \right) \end{array} \right)$$

No Wilson lines. Gauge group $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$.

sing. fibers of $\bar{\mathbf{X}}$: $2\,II^* \oplus 4\,I_1$,

$\mathrm{NS}(\mathbf{X}) = H \oplus E_8 \oplus E_8$, signature: $(1, 17)$,

$\mathrm{T}_{\mathbf{X}} = H^2$, signature: $(2, 2)$.

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$c = 0$: No Wilson lines. Gauge group $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$.

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$c \neq 0$: One Wilson line. Gauge group $G = E_8 \times E_7$.

$$\begin{array}{lcll} \text{sing. fibers of } \bar{\mathbf{X}} & : & II^* \oplus III^* \oplus 5\,I_1\,, & \\ \mathrm{NS}(\mathbf{X}) & = & H \oplus E_8 \oplus E_7\,, & \text{signature: } (1,16)\,, \\ \mathrm{T}_{\mathbf{X}} & = & H^2 \oplus \langle -2 \rangle\,, & \text{signature: } (2,3)\,. \end{array}$$

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For $G = \mathrm{Spin}(32)/\mathbb{Z}_2$ *and* $G = \mathrm{Spin}(28) \times \mathrm{SU}(2)/\mathbb{Z}_2$*:* $\bar{\mathbf{X}} \dashrightarrow \bar{\mathbf{X}}_{\mathrm{alt}}$

Shioda-Inose correspondence

Def.: A Nikulin involution on a K3 surface $\mathbf{X}$ is an analytic automorphism $\beta : \mathbf{X} \to \mathbf{X}$ of order two such that $\beta^* \eta = \eta$.

$\Rightarrow$ β has eight fixpoints, $\mathbf{Y} = \widetilde{\mathbf{X}/\beta}$ is K3 surface,
$\exists \, \mathrm{p} : \mathbf{X} \dashrightarrow \mathbf{Y}$ degree-two rational map, $\mathrm{p}_* : H^2(\mathbf{X}, \mathbb{Z}) \to H^2_{\mathbf{Y}}$;

Shioda-Inose correspondence

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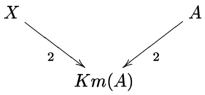
Def.: A Nikulin involution is a Shioda-Inose structure if there is an Abelian surface **A** such that $\mathbf{Y} = \mathrm{Kum}(\mathbf{A}) = \widetilde{\mathbf{A}/\{\pm 1\}}$ and $\mathrm{p}_* : \mathrm{T}_{\mathbf{X}}(2) \to \mathrm{T}_{\mathbf{Y}}$ is Hodge isometry.



Morrison '84: An algebraic K3 surface **X** *has a Shioda-Inose structure if there exists* **A** *and Hodge isometry* $\mathrm{T}_{\mathbf{X}} \cong \mathrm{T}_{\mathbf{A}}$.

Resulting picture for heterotic/F-theory duality

(based on Clingher-Doran '07, '10; A. Kumar '08)

c/z	F-theory moduli for $\bar{\mathbf{X}} \to \mathbb{CP}^1$	heterotic moduli for $\mathbf{A}$
$= 0$	$\mathrm{T}_{\mathbf{X}} = \mathrm{T}_{\mathbf{A}} = H^2$	
	$\mathrm{NS}(\mathbf{X}) = H \oplus E_8 \oplus E_8$ $a \simeq E_4(\rho)\, E_4(\sigma)$ $b \simeq E_6(\rho)\, E_6(\sigma)$ $d \simeq \eta^{24}(\rho)\, \eta^{24}(\sigma)$	$\mathbf{A} = E_\rho \times E_\sigma$ $E_\rho : \; [E_4(\rho) : E_6(\rho)] \in \mathbb{WP}_{(4,6)}$ $E_\sigma : \; [E_4(\sigma) : E_6(\sigma)]$
	$(\rho, \sigma) \in \Gamma \backslash \mathrm{SO}(2,2) / \big(\mathrm{SO}(2) \times \mathrm{SO}(2)\big) = \Gamma \backslash \mathbb{H} \times \mathbb{H}$	
$\neq 0$	$\mathrm{T}_{\mathbf{X}} = \mathrm{T}_{\mathbf{A}} = H^2 \oplus \langle -2 \rangle$	
	$\mathrm{NS}(\mathbf{X}) = H \oplus E_8 \oplus E_7$ $a \simeq \mathcal{E}_4(\underline{\tau})\,, \quad b \simeq \mathcal{E}_6(\underline{\tau})\,,$ $c \simeq \mathcal{C}_{10}(\underline{\tau})\,, \quad d \simeq \mathcal{C}_{12}(\underline{\tau})\,.$	$\mathbf{A} = \mathrm{Jac}\, C^{(2)}_{\underline{\tau}}$ $[I_2 : I_4 : I_6 : I_{10}] \in \mathbb{WP}^3_{(2,4,6,10)}$ $I_2 \simeq \frac{\mathcal{C}_{12}}{\mathcal{C}_{10}},\; I_4 \simeq \psi_4,$ $I_6 \simeq \mathcal{E}_6 + \frac{\mathcal{E}_4\, \mathcal{C}_{12}}{\mathcal{C}_{10}},\; I_{10} \simeq \mathcal{C}_{12}$
	$\underline{\tau} = \begin{pmatrix} \rho & z \\ z & \sigma \end{pmatrix} \in \Gamma \backslash \mathrm{SO}(2,3) / \big(\mathrm{SO}(2) \times \mathrm{SO}(3)\big) = \mathrm{Sp}(4, \mathbb{Z}) \backslash \mathbb{H}_2$	

Calabi-Yau threefolds and 6d compactifications

	$\mathrm{T}_{\mathbf{X}} = \mathrm{T}_{\mathbf{A}} = H^2 \oplus \langle -2 \rangle$	
$\rho \neq \sigma$ $z \neq 0$	$\mathrm{NS}(\mathbf{X}) = H \oplus E_8 \oplus E_7$ $a \simeq \mathcal{E}_4(\underline{\tau}), \quad b \simeq \mathcal{E}_6(\underline{\tau}),$ $c \simeq \mathcal{C}_{10}(\underline{\tau}), \quad d \simeq \mathcal{C}_{12}(\underline{\tau}).$	$\mathbf{A} = \mathrm{Jac}\, C^{(2)}_{\underline{\tau}}$ $[I_2 : I_4 : I_6 : I_{10}] \in \mathbb{WP}^3_{(2,4,6,10)}$ $I_2 \simeq \frac{\mathcal{C}_{12}}{\mathcal{C}_{10}}, \; I_4 \simeq \psi_4,$ $I_6 \simeq \mathcal{E}_6 + \frac{\mathcal{E}_4\, \mathcal{C}_{12}}{\mathcal{C}_{10}}, \; I_{10} \simeq \mathcal{C}_{12}$
	$\underline{\tau} = \begin{pmatrix} \rho & z \\ z & \sigma \end{pmatrix} \in \Gamma \backslash \mathrm{SO}(2,3) / \Big(\mathrm{SO}(2) \times \mathrm{SO}(3)\Big) = \mathrm{Sp}(4,\mathbb{Z}) \backslash \mathbb{H}_2$	

$$\begin{array}{ccc} E & \rightarrow & \bar{\mathbf{X}} \\ & & \downarrow \\ [t{:}1] \in \mathbb{CP}^1 & \rightarrow & \mathbb{F}_{12} \\ & & \downarrow \\ & & \mathbb{CP}^1 \ni [u{:}1] \end{array}$$

$$\begin{aligned} Y^2 \;=\; & 4\, X^3 + \Big(a(u)\, t^4 + c(u)\, t^3\Big)\, X \\ & + \Big(t^7 + b(u)\, t^6 + d(u)\, t^5\Big) \end{aligned}$$

$$a \in H^0(\mathcal{O}(8)),\, b \in H^0(\mathcal{O}(12)),\, etc.$$

M.-Morrison '13: *Construction of smooth Calabi-Yau threefolds* $\mathbf{X} \rightarrow \mathbf{F}_{12}$ *from pencils of genus-2 curves confirms existence heterotic quantum vacua and shows gauge group enhancement of* $\mathbf{X}_u$ *when family intersects* $H_1 + H_4$.

$\mathcal{N} = 2$ String Compactification

- 4d string compactifications arise in two ways (that are dual)
  1. from F-theory on n-dimensional K3-fibered Calabi-Yau,
  2. from heterotic strings on $(n-1)$-dimensional elliptically fibered Calabi-Yau.

- Four-dimensional string compactifications:
  include Donaldson theory/Seiberg-Witten theory of M^4

- Effective field theory for 4d $\mathcal{N} = 2$ supersymmetric Yang-Mills theory can be described in terms of auxiliary family of elliptic curves, called SW-curve (=rational elliptic surface).

$\mathcal{N} = 2$ String Compactification

- **Sen ['95]** studied F-theory over a K3-surface in the special point where K3 is the $\mathbb{Z}_2$-orbifold of torus T^4.
- Sen provided an embedding in F-theory of SW-curve.
- In isotrivial case: SW-curve is rational elliptic surface with $2\, I_0^*$, the embedding was given by quadratic twist K3 with $4 I_0^*$ (doesn't change j-invariant)
- Masses of BPS states were computed in F-theory in terms of period integrals of the holomorphic 2-form on the K3 surface.
- **Goal:** what Weierstrass elliptic K3 surfaces (=F-theory vacua) are obtained when generalizing construction to rational elliptic surfaces? what are the corresponding heterotic vacua?

Rational surfaces

- Rational Weierstrass elliptic surfaces **S** over $\mathbb{CP}^1$:

$$\bar{\mathbf{S}}: \; y^2 = 4\,x^3 - g_2\,x - g_3\,, \qquad \begin{array}{l} g_2 \in H^0(\mathcal{O}(4)), \\ g_3 \in H^0(\mathcal{O}(6)), \end{array} \quad [t:1] \in \mathbb{CP}^1.$$

- Consider rational elliptic surfaces with at most 3 singular fibers, $(\#\mathrm{sings}, \mathrm{rk}(\mathrm{MW})) = (2,0), (3,0), (3,1), (3,2)$.

Examples:

- SW-curve **S** for pure $\mathrm{SU}(2)$-gauge theory:
  Legendre family over the t-line,
  t Hauptmodul for $\Gamma(2)$,
  $y^2 = x(x-1)(x-t)$

E_{sing}	I_2	I_2	$I_2^*(= D_6)$
t	0	1	∞

- Pencil related by 2-isogeny

E_{sing}	I_1	I_1	$I_4^*(= D_8)$
t	0	1	∞

Rational surfaces

Examples (cont'd):

- Hesse pencil in $\mathbb{P}(1,1,1)$ over the t-line,
  t^3 Hauptmodul for $\Gamma_0(3)$,
  $x_1^3 + x_2^3 + x_3^3 + t^{-1/3}\, x_1\, x_2\, x_3 = 0$

E_{sing}	I_1	I_3	$IV^*(= E_6)$
t	0	$\frac{1}{27}$	∞

- Hypersurfaces in $\mathbb{P}(1,1,2)$ over the t-line,
  t^4 Hauptmodul for $\Gamma_0(2)$,
  $x_1^4 + x_2^4 + x_3^2 + t^{-1/4}\, x_1\, x_2\, x_3 = 0$

E_{sing}	I_1	I_2	$III^*(= E_7)$
t	0	$\frac{1}{64}$	∞

- Hypersurfaces in $\mathbb{P}(1,2,3)$ over the t-line,
  $x_1^6 + x_2^3 + x_3^2 + t^{-1/6}\, x_1\, x_2\, x_3 = 0$

E_{sing}	I_1	I_1	$II^*(= E_8)$
t	0	$\frac{1}{432}$	∞

Rational surfaces and their periods

- Rational Weierstrass elliptic surfaces **S** ($\#$sings ≤ 3)

$$\bar{\mathbf{S}} : \; y^2 = 4\,x^3 - g_2\,x - g_3\,, \qquad \begin{array}{l} g_2 \in H^0(\mathcal{O}(4)), \\ g_3 \in H^0(\mathcal{O}(6)), \end{array} \quad [t:1] \in \mathbb{CP}^1.$$

- Rational surfaces (up to $*$-transfer):

isotrivial		
I_0	I_0^*	I_0^*
I_0	IV	IV^*
I_0	III	III^*
I_0	II	II^*
IV	IV	IV
II	II	IV^*

modular		
I_1	I_1	I_4^*
I_2	I_2	I_2^*
I_3	I_1	IV^*
I_2	I_1	III^*
I_1	I_1	II^*
II	I_2	IV^*
II	I_1	III^*

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isotrivial			G	r
I_0	I_0^*	I_0^*	$\mathbb{Z}_2$	0
I_0	IV	IV^*	$\mathbb{Z}_3$	0
I_0	III	III^*	$\mathbb{Z}_4$	0
I_0	II	II^*	$\mathbb{Z}_6$	0
IV	IV	IV	$\mathbb{Z}_3$	2
II	II	IV^*	$\mathbb{Z}_6$	2

modular			G	r
I_1	I_1	I_4^*	$\Gamma_0(4)$	0
I_2	I_2	I_2^*	$\Gamma(2)$	0
I_3	I_1	IV^*	$\Gamma_0(3)$	0
I_2	I_1	III^*	$\Gamma_0(2)$	0
I_1	I_1	II^*	Γ	0
II	I_2	IV^*	Γ^2	1
II	I_1	III^*	Γ	1

$G \subset \mathrm{SL}(2,\mathbb{Z})$ generated monodromy group, $r = \mathrm{rk}(\mathrm{MW}^0)$

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modular			G	r
I_1	I_1	I_4^*	$\Gamma_0(4)$	0
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I_3	I_1	IV^*	$\Gamma_0(3)$	0
I_3	I_1	III^*	$\Gamma_0(2)$	0
I_1	I_1	II^*	Γ	0
II	I_2	IV^*	Γ^2	1
II	I_1	III^*	Γ	1

- Write down Picard-Fuchs first order linear system satisfied by periods of $\frac{dx}{y}$ and $\frac{x\,dx}{y}$ over cycles on the fibers:

$$\vec{u}^t = \left(\omega = \int_{A_t} \frac{dx}{y},\, a = \int_{A_t} \frac{x\,dx}{y}\right)$$

Rational surfaces and their periods

- Rational Weierstrass elliptic surfaces **S** (#sings ≤ 3)

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- Rational surfaces (up to $*$-transfer):

isotrivial			G	r
I_0	I_0^*	I_0^*	$\mathbb{Z}_2$	0
I_0	IV	IV^*	$\mathbb{Z}_3$	0
I_0	III	III^*	$\mathbb{Z}_4$	0
I_0	II	II^*	$\mathbb{Z}_6$	0
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modular			G	r
I_1	I_1	I_4^*	$\Gamma_0(4)$	0
I_2	I_2	I_2^*	$\Gamma(2)$	0
I_3	I_1	IV^*	$\Gamma_0(3)$	0
I_2	I_1	III^*	$\Gamma_0(2)$	0
I_1	I_1	II^*	Γ	0
II	I_2	IV^*	Γ^2	1
II	I_1	III^*	Γ	1

- Defines a rank-two Fuchsian system with flat (=integrable) meromorphic connection over base:

$$\frac{d}{dt}\vec{u} = \mathbf{A}(t)\,\vec{u}\,, \qquad \mathbf{A}(t) = \frac{\mathbf{B}_0(t)}{t} + \frac{\mathbf{B}_1(t)}{t-1}\,, \qquad \sum_i \mathbf{B}_i(t) = 0\,.$$

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- Rational Weierstrass elliptic surfaces **S** (#sings ≤ 3)

$$\bar{\mathbf{S}}: \; y^2 = 4\,x^3 - g_2\,x - g_3\,, \qquad \begin{array}{l} g_2 \in H^0(\mathcal{O}(4)), \\ g_3 \in H^0(\mathcal{O}(6)), \end{array} \quad [t:1] \in \mathbb{CP}^1 .$$

- Rational surfaces (up to $*$-transfer):

isotrivial			μ	κ
I_0	I_0^*	I_0^*	$1/2$	0
I_0	IV	IV^*	$1/3$	0
I_0	III	III^*	$1/4$	0
I_0	II	II^*	$1/6$	0
IV	IV	IV	$1/3$	$2/3$
II	II	IV^*	$1/6$	$2/3$

modular			μ	κ
I_1	I_1	I_4^*	$1/2$	0
I_2	I_2	I_2^*	$1/2$	0
I_3	I_1	IV^*	$1/3$	0
I_2	I_1	III^*	$1/4$	0
I_1	I_1	II^*	$1/6$	0
II	I_2	IV^*	$1/6$	$1/3$
II	I_1	III^*	$1/12$	$1/3$

Solutions to Picard-Fuchs rank-2 first order linear system:

$$\omega = t^{\mp 3\kappa\mu/2}\; {}_1F_0(\pm\mu;\,|t) \qquad \omega = t^{-\kappa/2}\; {}_2F_1(\mu, 1-\mu-\kappa; 1-\kappa|t)$$

One-parameter families of K3 surfaces

- Construction 1: quadratic twist with polynomial h

$$\begin{aligned} \bar{\mathbf{X}}_1 = \bar{\mathbf{S}}_h : \; Y^2 &= 4\,X^3 - h^2\, g_2\, X - h^3\, g_3 \\ &\downarrow \\ \bar{\mathbf{S}} : \; y^2 &= 4\,x^3 - g_2\, x - g_3 \; . \end{aligned}$$

- Twist around $t = \infty$ introduces fibers of type I_0^*
- Parameter defines position of additional I_0^*, $h{=}t(t{-}A)$
- 1-parameter families of lattice-polarized K3 surfaces, isotrivial/modular case: Picard rank 18/19
- Example: $\mathrm{T}_{\mathbf{X}} = \langle 2 \rangle^{\oplus 2} \oplus \langle -2 \rangle$, $A \notin \{0,1\}$:

E_{sing}	I_2	I_2	I_2^*
t	0	1	∞

$\mathbf{S}$ is rational

E_{sing}	I_2^*	I_2	I_2^*	I_0^*
t	0	1	∞	A

$\mathbf{X}_1$ is K3

- For $A = 1$ we obtain all Weierstrass elliptic K3 surfaces with 3 singular fibers.

One-parameter families of K3 surfaces

- Construction 1: quadratic twist with polynomial h

$$\begin{array}{rcl} \bar{\mathbf{X}}_1 = \bar{\mathbf{S}}_h : \; Y^2 & = & 4\,X^3 - h^2\,g_2\,X - h^3\,g_3 \\ & \downarrow & \\ \bar{\mathbf{S}} : \; y^2 & = & 4\,x^3 - g_2\,x - g_3\,. \end{array}$$

- 2 I_0^*'s, $h{=}t(t{-}A)$, 2-form: $dt \wedge \frac{dX}{Y} = \frac{1}{\sqrt{h(t)}}\, dt \wedge \frac{dx}{y}$
- Represent K3-periods as Euler transforms $\int_{\gamma_i} dt \wedge \frac{dX}{Y} = \int_{0^*}^{t_i^*} dt\, \frac{1}{\sqrt{h(t)}}\, \omega$
- K3-periods solve a rank-4/rank-3 linear system in ∂_A

Solutions to the rank-4/rank-3 integrable linear system of K3 periods:

$$\omega = \begin{array}{c} {}_2F_1\left(\frac{1}{2}, \mu{+}\frac{\kappa}{2}; 1{+}\frac{\kappa}{2} \middle| A\right) \\ \oplus \\ {}_2F_1\left(\frac{1}{2}, -\mu{-}\frac{\kappa}{2}; 1{-}\frac{\kappa}{2} \middle| A\right) \end{array} \qquad \omega = A^{-\kappa/2}\; {}_3F_2\left(\begin{array}{c} \mu, \frac{1-\kappa}{2}, 1{-}\mu{-}\kappa \\ 1{-}\frac{\kappa}{2}, 1{-}\kappa \end{array} \middle| A \right)$$

One-parameter families of K3 surfaces

Proposition (M.-Doran)

- *The periods of the seven (modular) families of Weierstrass elliptic K3 surfaces of Picard rank 19 satisfy Clausen's identity:*

$$A^{-\kappa/2}\ {}_3F_2\left(\begin{array}{c} \mu,\ \frac{1-\kappa}{2},\ 1-\mu-\kappa \\ 1-\frac{\kappa}{2},\ 1-\kappa \end{array}\middle| A\right) = \left(A^{-\kappa/4}\ {}_2F_1\left(\tfrac{\mu}{2},\ \tfrac{1-\mu-\kappa}{2};\ 1-\tfrac{\kappa}{2}\middle| A\right)\right)^2$$

- *There is a fundamental set of solutions* $\{x_1, x_2, x_3\}$ *such that*

μ	*quadric surface*	*series*
$1/2$	$x_1^2 + x_2^2 - x_3^2$ $2\,x_1^2 + 2\,x_2^2 - 2\,x_3^2$	${}_3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array}\middle| A\right) = \sum_{n=0}^{\infty} \frac{(2n)!^3}{n!^6} \frac{A^n}{2^{6n}}$
$1/3$	$4\,x_1^2 + 3\,x_2^2 - 3\,x_3^2$	${}_3F_2\left(\begin{array}{c} \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \\ 1, 1 \end{array}\middle| A\right) = \sum_{n=0}^{\infty} \frac{(2n)!\,(3n)!}{n!^5} \frac{A^n}{2^{2n}3^{3n}}$
$1/4$	$4\,x_1^2 + 2\,x_2^2 - 2\,x_3^2$	${}_3F_2\left(\begin{array}{c} \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \\ 1, 1 \end{array}\middle| A\right) = \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{A^n}{4^{4n}}$
$1/6$	$x_1^2 + 4\,x_2^2 - x_3^2$	${}_3F_2\left(\begin{array}{c} \frac{1}{6}, \frac{3}{6}, \frac{5}{6} \\ 1, 1 \end{array}\middle| A\right) = \sum_{n=0}^{\infty} \frac{(6n)!}{n!^3(3n)!} \frac{A^n}{2^{6n}3^{3n}}$

One-parameter families of K3 surfaces

- Construction 2: double cover branched at $t=0$ and $t=A$:

$$\begin{array}{rcl} \bar{\mathbf{X}}_2 = \bar{\mathbf{S}}_{[0,A]}: \; Y^2 & = & 4\,X^3 - s^4\, g_2(t(s))\,X - s^6\, g_3(t(s)) \\ & \downarrow & \\ \bar{\mathbf{S}}: \; Y^2 & = & 4\,X^3 - g_2(t)\,X - g_3(t)\,. \end{array}$$

- with $t = \frac{(s+A/4)^2}{s}$ we have $ds \wedge \frac{dX}{Y} = \frac{1}{\sqrt{t(t-A)}}\, dt \wedge \frac{dx}{y}$
- Example of 1-param. family of lattice-polarized K3 surface of Picard rank 19, $\mathrm{T}_{\mathbf{X}} = H \oplus \langle -2 \rangle$, $A \notin \{0,1\}$:

E_{sing}	I_1	I_1	II^*
t	0	1	∞

S is rational

E_{sing}	I_2	$2\,I_1$	$2\,II^*$
s	$A/4$	$\frac{A}{4} + \frac{1}{2} \pm \sqrt{A+1}$	$0, \infty$

$\mathbf{X}_2$ is K3

- for $\mu = 1/6, 1/4, 1/3, 1/2, \kappa = 0$ and $\mu = 1/12, \kappa = 1/3$ we obtain one-parameter families with $M_n = H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle$ lattice polarization for $n = 1, 2, 3, 4$ and $n = 6$.

One-parameter families of K3 surfaces

Proposition (M.-Doran)

- *The two constructions give rise to degree-two rational maps* $\mathbf{X}_2 \dashrightarrow \mathbf{X}_1$ *(for all 13 cases) that leave the holomorphic two-form invariant.*
- *The Picard-Fuchs differential equations of each pair* $\mathbf{X}_2, \mathbf{X}_1$ *coincide.*

Remarks:

- The periods of the families with M_n lattice polarization for $n = 1, 2, 3, 4$ and $n = 6(?)$ agree with the results of **Lian, Yau ['96]**, **Dolgachev ['96]**, **Verrill, Yui['00]**, **Doran ['00]**, and **Beukers, Peters ['84]** (?).
- Constructions generalize to **two**-parameter families of lattice-polarized Weierstrass elliptic K3 surfaces: isotrivial/modular case for $\kappa = 0$: Picard rank 16/18.

Two-parameter families of K3 surfaces

Set $h(t) = (t - A)(t - B)$ in $\mathbf{X}_1$ and $t = \frac{16\, s^2 + 8\, (A+B)\, s + (A-B)^2}{16\, s}$ in $\mathbf{X}_2$ s.t.

$$ds \wedge \frac{dX}{Y} = \frac{1}{\sqrt{h(t)}}\, dt \wedge \frac{dx}{y}$$

Proposition (M.-Doran)

- *The two constructions give rise to degree-two rational maps* $\mathbf{X}_2 \dashrightarrow \mathbf{X}_1$ *(for all cases with* $\kappa = 0$*) that leave the holomorphic two-form invariant.*
- *The Picard-Fuchs linear systems for each pair* $\mathbf{X}_2, \mathbf{X}_1$ *coincide.*
- *K3-periods solve a flat rank-6/rank-4 linear system in* ∂_A, ∂_B *in the isotrivial/modular cases.*

Two-parameter families of K3 surfaces

Proposition (M.-Doran)

- *The two constructions give rise to degree-two rational maps* $\mathbf{X}_2 \dashrightarrow \mathbf{X}_1$ *(for all cases with* $\kappa = 0$*) that leave the holomorphic two-form invariant.*
- *K3-periods solve a flat rank-6/rank-4 linear system in* ∂_A, ∂_B *in the isotrivial/modular cases.*
- *Isotrivial cases* ($\mu \neq \frac{1}{2}, \kappa = 0$)*:*

$$F_1\left(\mu; \frac{1}{2}, \frac{1}{2}; 1 \middle| A, B\right) \oplus F_1\left(-\mu; \frac{1}{2}, \frac{1}{2}; 1 \middle| A, B\right)$$

- *Modular cases* ($\kappa = 0$)*:*

$$\Omega_\mu(A,B) = \frac{1}{B^\mu}\, F_2\left(\mu; \frac{1}{2}, \mu; 1, 2\mu \middle| 1 - \frac{A}{B}, \frac{1}{B}\right)$$

Two-parameter families of K3 surfaces

Remarks:

- $F_1(\alpha;\beta,\beta';\gamma)$ and $F_2(\alpha;\beta,\beta';\gamma,\gamma')$ are the Appell hypergeometric functions in two variables.
- They satisfy equations of a linear system of rank 4 or 3:

$$A(1-A)\,F_{AA}+p_i(A,B)\,F_{AB}+\Big(\gamma\quad-(\alpha+\beta\;+1)\,A\Big)\,F_A-\beta\;\;B\,F_B-\alpha\,\beta\;\;F=0\,,$$

$$B(1-B)\,F_{BB}+p_i(B,A)\,F_{AB}+\Big(\gamma^{(')}-(\alpha+\beta'+1)\,B\Big)\,F_B-\beta'\,A\,F_A-\alpha\,\beta' F=0\,.$$

- Example ($\mu=1/6$): $M=H\oplus E_8\oplus E_8$-polarized case,

E_{sing}	$2\,I_1$	$2\,I_1$	$2\,II^*$
s	$t(s)=0$	$t(s)=1$	$0,\ \infty$

$\mathbf{X}_2$ is M-polarized K3

$\overset{s.l.}{\dashrightarrow}$

E_{sing}	$2\,I_1$	II^*	$2\,I_0^*$
t	$0,1$	∞	A,B

$\mathbf{X}_1$ is $\mathrm{Kum}(E_1\times E_2)$

- Examples realize elliptic fibrations $\mathfrak{J}_3,\mathfrak{J}_4,\mathfrak{J}_6,\mathfrak{J}_7,\mathfrak{J}_{11}$ on $\mathrm{Kum}(E_1\times E_2)$ from **Oguiso ['88]**.

Two-parameter families of K3 surfaces

Remarks:

- F_2 satisfies Quadratic Condition (cf. **Sasaski, Yoshida ['88]**): fundamental solutions (x_1, x_2, x_3, x_4) are quadrically related, solution surfaces $S \subset \mathbb{P}^3$ reduces to $\mathbb{P}^1 \times \mathbb{P}^1$.
- Clausen-type equation:

$$\frac{1}{B^\mu} F_2\left(\mu; \tfrac{1}{2}, \mu; 1, 2\mu \middle| 1 - \tfrac{A}{B}, \tfrac{1}{B}\right) = \frac{1}{(1-A-B)^\mu} \cdot \begin{array}{l} {}_2F_1\left(\frac{\mu}{2}, \frac{\mu+1}{2}; 1 \middle| x\right) \\ {}_2F_1\left(\frac{\mu}{2}, \frac{\mu+1}{2}; \mu + \frac{1}{2} \middle| y\right) \end{array}$$

where $x(1-y) = \left(\frac{A-B}{1-A-B}\right)^2$, $y(1-x) = \left(\frac{1}{1-A-B}\right)^2$.

- F_2 satisfies linear and quadratic transformations (symmetries) (generalizing transformations for ${}_2F_1$):
  linear: $\Omega_\mu(A, B) = \Omega_\mu(B, A)$

Two-parameter families of K3 surfaces

Remarks:

- F_2 satisfies linear and quadratic transformations (symmetries) (generalizing transformations for ${}_2F_1$):

  quadratic: $$\Omega_{1/2}(A,B) = \left(\frac{2\,B}{1-A-B}\right)^{1/2} \Omega_{1/4}(\tilde{A},\tilde{B})$$

  $$\text{with } \tilde{A} = \left(\tfrac{A-B+1}{A+B-1}\right)^2, \tilde{B} = \left(\tfrac{A-B-1}{A+B-1}\right)^2.$$

- If we specialize $A = (\lambda/4)^2$, $B = 1 + A$ then we obtain

  $$\Omega_{1/2}(A,B) = \Omega_{1/4}\left(0, \left(\tfrac{4}{\lambda}\right)^4\right) = {}_3F_2\left(\begin{array}{c} \frac{1}{4},\frac{2}{4},\frac{3}{4} \\ 1,1 \end{array} \middle| \left(\tfrac{4}{\lambda}\right)^4\right)$$

  for the period of the sub-family $\mu = \frac{1}{2}, \kappa = 0$ (agrees with **Narumiya, Shiga ['01]**) which is birational to

  $$\mathcal{F} = \left\{xyz\,(x+y+z+\lambda) + 1 = 0\right\} \subset \mathbb{P}^3 ,$$

Periods of 3-parameter families of K3 surfaces

- There is only <u>one</u> family where the construction of $\mathbf{X}_1$ can be turned into a 3-parameter family of K3 surfaces with lattice polarization of Picard-Rank 17: $\mu = \frac{1}{2}, \kappa = 0$.
- Use $h(t) = (t - A)(t - B)(t - C)$ to obtain linear system of rank 5 in A, B, C for the K3-periods on $\mathbf{X}_1$ = specialization of Aomoto-Gel'fand HGF of type (3,6)

$$E(3,6)\left(\alpha_i = \frac{1}{2}\Big| u, v, 0, w\right)$$

where $u = \left(\frac{C-A}{B-A}\right)\frac{B}{C}, v = \frac{B}{C}, w = B$.

- Linear system specialization of the one in **Matsumoto et. al ['93]** for a family of K3 surfaces of Picard rank 16 associated with six lines in the complex plane, no three of which are concurrent.

Kummer surfaces from $SU(2)$-Seiberg-Witten curve

Proposition (M.-Doran)

- *The family* $\mathbf{X}_1 = \mathbf{S}_h \to \mathbb{CP}^1$ $(\mu = 1/2, \kappa = 0)$ *is a family of Jacobian K3 surfaces with* N*-polarization and* $\mathrm{N}^\perp = H^2(2) \oplus \langle -2 \rangle$.
- *There is a family* $\mathbf{X}_2 \to \mathbb{CP}^1$ *obtained from the covering map* $t = (C\,s^2 - B)/(s^2-1)$, $g_i(t) \mapsto g_i(t(s))\, h^i(t(s))\ \left((s^2-1)^2/s\right)^{2i}$.
- $\mathbf{X}_2 = \mathrm{Kum}(\mathbf{A})$ *where*

ρ	*parameter*	**A**	*equation*	*moduli space*
17	u,v,w	$\mathrm{Jac}\,C^{(2)}$	$y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$	$\Gamma_2(2)\backslash\mathbb{H}_2$
18	$u,w,v=0$	$E_1 \times E_2$	$y_i^2 = (2x_i - 1)\left[(4x_i+1)^2 + 9r_i\right]$	$\Gamma\backslash\mathbb{H}\times\mathbb{H}$
19	$u,v=0,w=1$	$E_1 \times E_1$	$y_1^2 = (2x_1 - 1)\left[(4x_1+1)^2 + 9r_1\right]$	$\Gamma_0(2)\backslash\mathbb{H}$

Mayr, Stieberger ['95], Kokorelis ['99]: moduli space of genus-two curves with level-two structure = moduli space of $\mathcal{N}=2$ heterotic string theories compactified on $K3 \times T^2$ with one Wilson line.