Self-Shuffling Words

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Challenges in Combinatorics on Words, Toronto, April 2013

A finite word w is obtained as a shuffle of two words x and y if it can be factorized as

$$w = \prod_{i=1}^{n} a_i b_i$$

with

$$x = \prod_{i=1}^{n} a_i \quad \text{and} \quad y = \prod_{i=1}^{n} b_i .$$

- Can x be obtained as a shuffle of a word y with itself?

Definition

An infinite word $x \in \mathbb{A}^{\mathbb{N}}$ is self-shuffling if it admits the factorizations:

$$x = \prod_{i=1}^{+\infty} U_i V_i = \prod_{i=1}^{+\infty} U_i = \prod_{i=1}^{+\infty} V_i$$

for some $U_i, V_i \in \mathbb{A}^+$.

Self-shuffling and periodicity

- Every purely periodic word $x = u^\omega$ is self-shuffling:

$$x = \prod_{i=1}^{+\infty}(uu) = \prod_{i=1}^{+\infty} u.$$

- Every letter of a self-shuffling word must occur an infinite number of times.
- Hence the word 01^ω is not self-shuffling.
- The word $0110(0011)^\omega$ is self-shuffling but $011(10)^\omega$ is not.

Fibonacci word

- The Fibonacci word $F = F_1 F_2 F_3 \cdots$, where $F_i \in \{0, 1\}$, is self-shuffling:

$$F = \prod_{i=1}^{+\infty} \sigma(F_i) F_i = \prod_{i=1}^{+\infty} \sigma(F_i) = \prod_{i=1}^{+\infty} F_i$$

where $\sigma \colon 0 \mapsto 01; 1 \mapsto 0$.

It suffices to observe that $\sigma^2(0) = \sigma(0)0$ and $\sigma^2(1) = \sigma(1)1$.

$$F = \underbrace{0\,1\,0}_{\sigma^2(0)}\ \underbrace{0\,1}_{\sigma^2(1)}\ \underbrace{0\,1\,0}_{\sigma^2(0)}\ \underbrace{0\,1\,0}_{\sigma^2(0)}\ \underbrace{0\,1}_{\sigma^2(1)}\ \underbrace{0\,1\,0}_{\sigma^2(0)}\ \underbrace{0\,1}_{\sigma^2(1)}\ \underbrace{0\,1\,0}_{\sigma^2(0)}\ \underbrace{0\,1\,0}_{\sigma^2(0)} \cdots$$

- The morphic image of a self-shuffling word is again self-shuffling:

  If

  $$x = \prod_{i=1}^{+\infty} U_i V_i = \prod_{i=1}^{+\infty} U_i = \prod_{i=1}^{+\infty} V_i$$

  then

  $$\mu(x) = \prod_{i=1}^{+\infty} \mu(U_i)\mu(V_i) = \prod_{i=1}^{+\infty} \mu(U_i) = \prod_{i=1}^{+\infty} \mu(V_i).$$

- Can be used as a tool for showing that a word is not the morphic image of another word.

A necessary condition

A finite word is Abelian border-free if no proper suffix is Abelian equivalent to a proper prefix.

- A self-shuffling word must begin in only finitely many Abelian border-free prefixes.

$x =$ 

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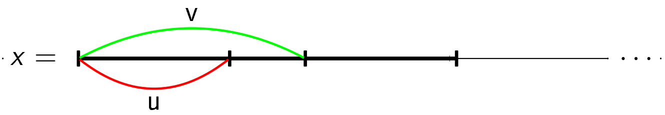

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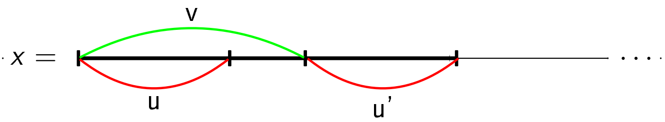

- The word $0F$ is not self-shuffling:

  We know F begins in arbitrarily large palindromes B followed by 1.

  Hence $0F$ begins in arbitrarily large words of the form $0B1$.

  Those words $0B1$ are all Abelian border-free.

- The paper-folding word is not self-shuffling:

0_1_0_1_0_1_0_1_0_1_0_1_0_1_0_1 ⋯

001_011_001_011_001_011_001_011 ⋯

0010011_0011011_0010011_0011011 ⋯

001001100011011_001001110011011 ⋯

0010011000110110001001110011011 ⋯

One can check that all prefixes of length $2^j - 1$ are Abelian border-free:

$$0,\ 001,\ 0010011,\ 001001100011011, \ldots$$

Main results - Lyndon words

An infinite word x is Lyndon if $x <_{\text{lex}} T^n(x)$ for all $n \geq 1$.

Theorem 1

Let x and y be in the shift orbit closure of an infinite word z, and suppose that x is Lyndon. Then, if w can be obtained as a shuffle of x and y, then $w <_{\text{lex}} y$.

In particular, Lyndon infinite words are never self-shuffling.

Consequences:

- This gives another proof that $0F$ is not self-shuffling since it is Lyndon.
- It is well known that the first shift

$$x = 110100110010110 \cdots$$

of the Thue-Morse word is Lyndon, and hence is not self-shuffling. Yet it can be verified that x begins in only a finite number of Abelian border-free words.

Thue-Morse word

Theorem 2

The Thue-Morse word $\mathbf{T} = 011010011001\cdots$ fixed by the morphism $\tau: 0 \mapsto 01;\ 1 \mapsto 10$ is self-shuffling.

$$\mathbf{T} = 0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,1\,0\,0\,1\,0\,1\,1\,0\,0\,1\,1\,0\,1\,\cdots$$

Sturmian words

Here $\rho(x)$ designates the intercept of the Sturmian word x.

Theorem 3

Let S, M and L be Sturmian words of the same slope satisfying $S \leq_{\text{lex}} M \leq_{\text{lex}} L$. Then M can be obtained as a shuffle of S and L iff the following conditions are verified:

- If $\rho(M) = \rho(S)$ then $\rho(L) \neq 0$
- If $\rho(M) = \rho(L))$ then $\rho(S) \neq 0$.

In particular (taking $S = M = L$), we obtain

Corollary

A Sturmian word $x \in \{0,1\}^{\mathbb{N}}$ is self-shuffling iff x is not of the form aC where $a \in \{0,1\}$ and C is a characteristic Sturmian word.

As an application, we recover the following result:

Theorem (Yasutomi 1999, Berthé-Ei-Ito-Rao 2007)

Let $x \in \{0,1\}^{\mathbb{N}}$ *be a characteristic Sturmian word. If* y *is a pure morphic word in the orbit of* x, *then* $y \in \{x, 0x, 1x, 01x, 10x\}$.

References

- Dane Henshall, Narad Rampersad and Jeffrey Shallit, *Shuffling and unshuffling*, BEATCS, **107** (2012), 131-142.
- Émilie Charlier, Teturo Kamae, Svetlana Puzynina and Luca Zamboni, *Self-shuffling words*, to appear in LNCS (ICALP 2013), preprint on `arxiv:1302.3844`.
- Romeo Rizzi and Stéphane Vialette, *On recognizing words that are squares for the shuffle product*, to appear in LNCS (CSR 2013)