Smoothness spaces on Ahlfors regular sets

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joint work with Antti Vähäkangas and Riikka Korte

University of Helsinki

Toronto – August 29, 2012

Sobolev spaces

For a non-negative integer k and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\mathbb{R}^n)$ consists of all functions $f \in L^p(\mathbb{R}^n)$ having distributional derivatives $D^j f$, $|j| \leq k$, in $L^p(\mathbb{R}^n)$.

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Besov spaces

Let $\alpha > 0$, $1 \le p, q \le \infty$ and k be the integer such that $0 \le k < \alpha \le k+1$. Then $B^{\alpha}_{p,q}(\mathbb{R}^n)$ consists of functions $f \in L^p(\mathbb{R}^n)$ such that

$$\sum_{|j|\le k} \|D^j f\|_p + \sum_{|j|=k} \left(\int_{\mathbb{R}^n} \frac{\|D^j f(\cdot + h) - D^j f(\cdot)\|_p^q}{|h|^{n+(\alpha-k)q}}\, dh \right)^{1/q} < \infty,$$

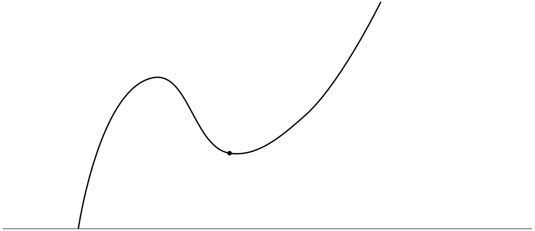
if $k < \alpha < k+1$ and $1 \le p, q < \infty$.
If $q = \infty$, then the usual interpretation in the limiting way is used.
If $\alpha = k+1$, the first difference of $D^j f$ is replaced by the second difference.

Characterization in terms of local polynomial approximations

Let $f \in L^u_{\text{loc}}(\mathbb{R}^n)$ and $1 \le u \le \infty$. The *normalized local best approximation* of f on a cube Q is

$$\mathcal{E}_k(f,Q)_{L^u(\mathbb{R}^n)} := \inf_{P \in \mathcal{P}_{k-1}} \left(\frac{1}{|Q|} \int_Q |f(x) - P(x)|^u \, dx \right)^{1/u},$$

where $\mathcal{P}_k$, $k \ge 0$ is a space of polynomials on $\mathbb{R}^n$ of degree at most k



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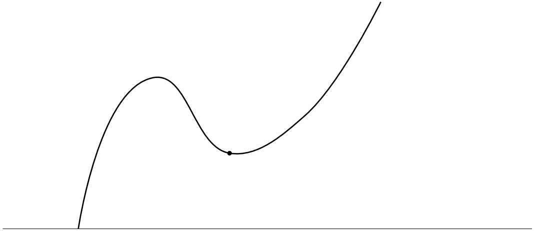

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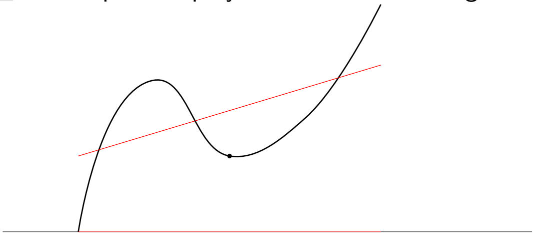

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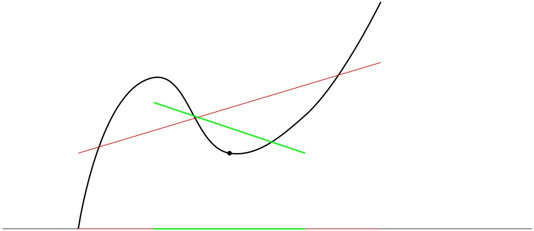

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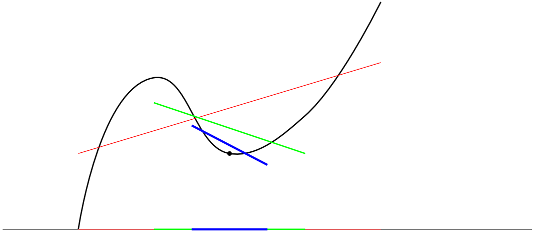

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The *sharp maximal function* of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is

$$f^\sharp_k(x) := \sup_{t>0} \frac{1}{t^k} \mathcal{E}_k(f, Q(x,t))_{L^1(\mathbb{R}^n)}, \; x \in \mathbb{R}^n.$$

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Let $p > 1$, $k \in \mathbb{N}$. Then Sobolev spaces

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$F^{\alpha}_{p,q}(\mathbb{R}^n)$ consists of functions $f \in L^p(\mathbb{R}^n)$ such that $g \in L^p(\mathbb{R}^n)$, where

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- H. Triebel - *Local approximation spaces*, Z. Anal. Anwendungen, 1989

Ahlfors d-regular sets

Let H^d denote d-dimensional Hausdorff measure on $\mathbb{R}^n$ and

$$Q(x,r) = \{y \in \mathbb{R}^n : \ \|x-y\|_\infty \le r\}.$$

A subset $S \subset \mathbb{R}^n$ is called an Ahlfors d-regular (or d-set) if there are c_1, $c_2 > 0$:

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- Examples
  Cantor-type sets,

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- The first approach to Besov spaces on d-sets, $0 < d \leq n$, in terms of jet functions
  A. Jonsson - *The trace of potentials on general sets*, Ark. Mat., 1979

Trace spaces on n-sets

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▷ P. Shvartsman - *Local approximations and intrinsic characterization of spaces of smooth functions on regular subsets of* $\mathbb{R}^n$, Math. Nachr., 2006

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Trace spaces on n-sets

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If Ω is a $W^{k,p}$-extension domain, $1 \leq p < \infty$, $k \geq 1$, then Ω is an n-set.

▷ P. Hajłasz, P.Koskela and H. Tuominen - *Sobolev embeddings, extensions and measure density condition*, J. Funct. Anal., 2008.

Trace of a function on a subset

Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $S \subset \mathbb{R}^n$. At those points $x \in S$ where exists

$$\bar{f}(x) = \lim_{r\to 0+} \fint_{Q(x,r)} f(y)\,dy = \lim_{r\to 0+} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} f(y)\,dy,$$

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$$d > n - \alpha p$$

the trace $f|_S$ is well defined.

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Let S be an d-set, $n-1<d<n$, $1\leq p,q<\infty$ and $\alpha>(n-d)/p$. Then

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- Trace theorems for Besov spaces and potential spaces on d-sets, $0<d<n$. A. Jonsson and H. Wallin *Function spaces on subsets of* $\mathbb{R}^n$, 1984

Traces of Triebel-Lizorkin spaces

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Remez-type inequality

Let S be a d-set, $n-1<d\le n$.

Suppose that $Q=Q(x_Q,r_Q)$ *and* $Q'=Q(x_{Q'},r_{Q'})$ *are cubes in* $\mathbb{R}^n$ *such that* $x_{Q'}\in S$, $Q'\subset Q$, $r_Q\le Rr_{Q'}$ *and* $r'_Q\le R$ *for some* $R>0$.

Then, $\forall\ p\in\mathcal{P}_k$

$$\left(\frac{1}{|Q|}\int_Q |p|^r\,dx\right)^{1/r}\le C\left(\frac{1}{\mathcal{H}^d(Q'\cap S)}\int_{Q'\cap S}|p|^u\,d\mathcal{H}^d\right)^{1/u},$$

where $1\le u,r\le\infty$ *and* C *depends on* S, R, n, u, r, k.

▷ A. Brudnyi and Yu. Brudnyi - *Remez type inequalities and Morrey-Campanato spaces on Ahlfors regular sets*, Contemp. Math., 2007

The construction of the extension operator is based on a modification of the Whitney extension method

Let $\mathcal{W}_S$ denote a Whitney decomposition of $\mathbb{R}^n \setminus S$ and

$\Phi := \{\varphi_Q : Q \in \mathcal{W}_S\}$ be a smooth partition of unity

To every cube $Q = Q(x_Q, r_Q) \in \mathcal{W}_S$ assign the cube $a(Q) := Q(a_Q, r_Q/2)$, where $a_Q \in S$ is such that $\|x_Q - a_Q\|_\infty = \operatorname{dist}(x_Q, S)$

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If $f \in L^1_{\mathrm{loc}}(S)$ and $k \in \mathbb{N}$, then

$$\operatorname{Ext}_{k,S} f(x) := \begin{cases} f(x), & \text{if } x \in S; \\ \sum_{Q \in \mathcal{W}_S} \varphi_Q(x) P_{k-1,Q} f(x), & \text{if } x \in \mathbb{R}^n \setminus S, \end{cases}$$

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where the projection $P_{k,Q} : L^1(a(Q) \cap S) \to \mathcal{P}_k$ are such that

$$\left(\fint_{a(Q)\cap S} |f - P_{k-1,Q} f|^u \, dH^d \right)^{1/u} \approx \mathcal{E}_k(f, a(Q))_{L^u(S)}.$$

Let $f \in L^u_{\mathrm{loc}}(S)$, $1 \le u \le \infty$, and Q be a cube centered at $S \subset \mathbb{R}^n$. Then *the normalized local best approximation* of f on Q in $L^u(S)$ norm is

$$\mathcal{E}_k(f,Q)_{L^u(S)} := \inf_{P \in \mathcal{P}_{k-1}} \left(\frac{1}{H^d(Q \cap S)} \int_{Q \cap S} |f - P|^u \, dH^d \right)^{1/u}.$$

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Fix $\alpha > 0$ and set $k = -[-\alpha]$. For $f \in L^u_{\text{loc}}(S)$ define *the fractional sharp maximal function*

$$f^\sharp_{\alpha,S}(x) := \sup_{t>0} \frac{1}{t^\alpha} \mathcal{E}_k(f, Q(x,t))_{L^1(S)}, \; x \in S,$$

and corresponding smoothness spaces

$$C^\alpha_p(S) = \{ f \in L^p(S) : \|f\|_{C^\alpha_p} = \|f\|_p + \|f^\sharp_{\alpha,S}\|_p < \infty \}, \quad p \ge 1.$$

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- (If $S = \mathbb{R}^n$) R. DeVore, R. Sharpley - *Maximal functions measuring local smoothness*, Memoirs of AMS, 1984

Theorem (R. Korte, L.I., 2011)

Let S *be an* d*-set with* $n-1<d\leq n$, $1<p\leq\infty$ *and* α *be a non-integer positive number. Then*

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- Relations between the trace spaces of *first order* Sobolev spaces and Hajłasz-Sobolev spaces
  P. Hajłasz and O. Martio - *Traces of Sobolev functions on fractal type sets and characterization of extension domains*, J. Funct. Anal., 1997

Are $C_k^p(S)$ relevant analogs of classical Sobolev spaces in the setting of an s-set?

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  ▷ P. Hajłasz and J.Kinnunen, - *Hölder quasicontinuity of Sobolev functions on metric spaces*, Rev. Mat. Iberoamericana, 1998.
- Let S be an s-set, $n-1 < s < n$, $p \geq 1$, $kp < s$ and $q = sp/(s-kp)$. Then

$$\|f\|_{L^q(S)} \leq c(\|f^\sharp_{\alpha,S}\|_{L^p(S)} + (\mathrm{diam} S)^{-\alpha}\|f\|_{L^p(S)})$$

Thank you for the attention