Extension of Sobolev Functions

to Capacitory Boundary

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ABSTRACT

Using the concept of the p-capacity, associated with uniform Sobolev spaces $L_p^1(\Omega)$, we introduce a notion of a p-capacitory boundary for an arbitrary domain $\Omega \subset \mathbb{R}^n$, $n-1 < p \leq n$. The p-capacitory boundaries depend on p and represent "ideal boundaries" of the domain Ω for p-capacitory metrics. The Sobolev classes $L_p^1(\Omega)$ can be extended to the p-capacitory boundaries under some additional assumptions on Ω.

The p-capacitory topology is equivalent to the Euclidean one into Ω, but the p-capacitory boundary depends on p and can be very far from the Euclidean one for non-regular domains. An analog of p-capacitory boundaries can be introduced for $p > n$ using a more delicate procedure.

For $p = n$ the notion of p-capacitory boundary was introduced by G. and Vodop'yanov in 80-th and was used for quasiconformal homeomorphisms extension on the n-capacitory boundary.

HISTORY and MOTIVATION

The concept of the ideal boundaries is common for geometry and analysis. The Poincare disc is a model of the hyperbolic plane that provides a geometrical realization of the ideal boundary of the hyperbolic plane with help of a conformal homeomorphism.

The two-dimensional theory of conformal homeomorphisms is very rich because the Riemann Mapping Theorem that states existence of conformal homeomorphism between the unit disc $B^2 \subset \mathbb{R}^2$ and any simply connected plane domain that has at least two boundary points.

However the boundary behavior of plane conformal homeomorphism can not be described in terms of Euclidean boundaries. The concept of ideal boundary elements (prime ends) was introduced by C. Caratheodory (1913) to describe boundary behavior of plane conformal homeomorphisms in geometric terms. The main Caratheodory theorem states that any conformal homeomorphism of unit disc induces one to one correspondence of prime ends.

M.A.Lavrentiev (1938) introduced a metric (a relative distance) for prime ends. G.D.Suvorov (1956) constructed a counterexample that demonstrates an absence of the triangle inequality for the Lavrentiev relative distance and proposed a more accurate concept of relative distance that is a metric. For this metric the Cartheodory prime ends represents a compactification of plane domains. There exists a number of different intrinsic conformally invariant metrics. A detailed survey can be found in a recent paper of V.M.Miklyukov .

There are two main attempts to construct a quasiconformally invariant "ideal" boundaries for dimension more than two. The first one is based on Royden algebras that are quasiconformal invariants by M. Nakai (1960} for dimension two and by L. G. Lewis (1971) for arbitrary dimension. As any Banach algebras the Royden algebras produces compactification of domains in space and any quasiconformal homeomorphisms induces a homeomorphism of the compactifications.

The second concept of so-called capacitory boundary was proposed by G. and S.Vodop'janov (1978) and is based on a notion of the conformal capacity. Remember that the conformal capacity is a quasiinvariant for quasiconformal homeomorphisms. Any quasiconformal homeomorphism can be extended to a homeomorphism of domains with capacitory boundaries. The concept of the capacitory boundary is based on capacitory metrics that will be discussed later.

The Royden compactification does not coincide with the Caratheodory compactification of two dimensional domains. The capacitory boundary coincides with the Caratheodory compactification.

What is connection between the prime ends theory and the theory of Sobolev spaces? Remember that the homogeneous Sobolev space $L^1_n(\Omega)$ are invariant under quasiconformal homeomorphisms. i.e. any quasiconformal homeomorphism φ between two domains $\Omega, \Omega' \in R^n$ induces an invertible bounded composition operator φ^* between spaces $L^1_n(\Omega')$ and $L^1_n(\Omega)$ (G., Vodop'janov, 1976).

Therefore the trace problem for homogeneous Sobolev spaces $L^1_n(\Omega)$ has direct connection with any capaitory compactification that is invariant under quasiconformal mappings

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set, then the uniform Sobolev space $L^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as

$$L^1_p(\Omega) = \{u \in L_{1,loc}(\Omega) | \nabla u \in L_p(\Omega)\}.$$

Here the gradient ∇u is in the weak (distributional) sense. The space $L^1_p(\Omega)$ is a seminormed space equipped with the seminorm

$$\|u | L^1_p(\Omega)\| = \|\nabla u | L_p(\Omega)\|.$$

The subset of C^∞-functions is dense in $L^1_p(\Omega)$.

The key role for the description of "‘ideal boundaries"’ plays a concept of p-capacity.

Recall that a well-ordered triple $E := (F_0, F_1; \Omega)$ of nonempty sets, where Ω is an open set in $\mathbb{R}^n$, and F_0, F_1 are compact subsets of $\overline{\Omega}$, is called a condenser on the Euclidean space $\mathbb{R}^n$.

The value

$$Cap_p(E) = Cap_p(F_0, F_1; \Omega) = \inf \int\limits_{\Omega} |\nabla v|^p \, dx, \; ,$$

where the infimum is taken over all nonnegative functions $v \in C(F_0 \cup F_1 \cup \Omega) \cap L^1_p(\Omega)$, such that $v \geq 0$ in a neighborhood of the set F_0, and $v \geq 1$ in a neighborhood of the set F_1, is called the p-capacity of the condenser $E = (F_0, F_1; \Omega)$.

Formally Sobolev functions defined only up to a set of measure zero, but they can be redefined pointwise up to a set of p-capacity zero. Indeed, $u \in L^1_p(\Omega)$ has a unique quasi-continuous representation, for which the function is continuous outside of an open subset of Ω with arbitrary small p-capacity. We use the concept of quasi-continuity for completion of a domain Ω with help of the p-capacity. Roughly speaking, an "ideal" p-capacitory boundary points are boundary continuums of p-capacity zero.

Our main theorem shows that for a large class of domains boundary values of functions $u \in L^1_p(\Omega)$, $n-1 < p \leq n$ on the p-capacitory boundary exist and the boundary values represents a function that is quasicontinuous on this "‘ideal boundary"’. For plane domains the 2-capacitory boundary coincides with the classical Caratheodory boundary.

For smooth domains traces of Sobolev. spaces are Besov spaces. In the case of Lipschitz domains the traces can be described also in terms of Besov spaces. For arbitrary non Lipschitz domain the problem is open. For cusp type singularities a description of traces can be found in (G.,Vasiltchik, 2010) in terms of weighted Sobolev spaces.

Fix a continuum F in the domain $\Omega \subset \mathbb{R}^n$ and a compact domain V such that $F \subset V \subset \overline{V} \subset \Omega$.

Definition 2.3. Choose arbitrarily points $x, y \in \Omega$ and joint x, y by a rectifiable curve $l(x, y)$. Define the p-capacitory quantity between x and y in Ω with respect to pair (F, V) as the value

$$\rho_{p;(F,V)}(x, y) = \inf_{l(x,y)} \{Cap_p^{\frac{1}{p}}(F, l(x, y) \setminus V; \Omega)+$$

$$Cap_p^{\frac{1}{p}}(\partial\Omega, l(x, y) \cap V; \Omega)\}$$

where the infimum is taken over all the curves $l(x, y)$.

We shall prove that the quantity $\rho_{p;(F,V)}(x,y)$ is an intrinsic metric in $\Omega \subset \mathbb{R}^n$ for $n-1 < p \leq n$. Note, that for $p \leq n-1$ and $p > n$ the p-capacitory quantity $\rho_{p;(F,V)}(x,y)$ is not a metric.

Denote by $\{\widetilde{\Omega}, \rho_{p;(F,V)}\}$ the standard completion of the metric space $\{\Omega, \rho_{p;(F,V)}\}$ and by H_p the set $\{\widetilde{\Omega}, \rho_{p;(F,V)}\} \setminus \left\{\Omega, \rho_{p;(F,V)}\right\}$.

We call H_p a p-capacitory boundary of Ω.

It will be proved that the topology of H_p, $n-1 < p \leq n$, does not depends on choice of a pair (F, V). Moreover two p-capacitory metrics are equivalent for any different choice of pairs (F_1, V_1) and (F_2, V_2). This is a justification of the notation H_p for the p-capacitory boundary.

The notion of a p-capacitory boundary was introduced for $p = n$ in 1982 (G.,Vodop'janov). It was proved the quasi-invariance of the n-capacitory metric under quasi-conformal homeomorphisms. For $p \neq n$ the situation is more complicated and interplay between p-quasi-conformal homeomorphisms and p-capacitory metrics will be discussed.

A homeomorphism φ between Euclidean domains Ω and Ω' is called p-quasiconformal (G.,Gurov,Romanov,1994) if φ is weakly differentiable and

$$|D\varphi(x)|^p \leq K|J(x,\varphi)|, \text{ for almost all } x \in \Omega.$$

In the case $p = n$ it is one of standard defini-topns of quasiconformal homeomorphisms.

Theorem 2 (G.,Gurov,Romanov)

The homeomorphism $\varphi : \Omega \to \Omega'$ is p-quasiconformal if and only if the mapping φ is weakly differentiable, is the mapping of finite distortion, and generates by the composition rule $\varphi^* f = f \circ \varphi$ a bounded composition operator on Sobolev spaces

$$\varphi^* : L^1_p(\Omega') \to L^1_p(\Omega), \; 1 \leq p < \infty.$$

Theorem 3 Let Ω be a domain in $\mathbb{R}^n$. Suppose $n-1 < p \leq n$ and $\rho_{p;(F_1,V_1)}$, $\rho_{p;(F_2,V_2)}$ are two p-capacitory quantities on Ω. Then there exist two constants $K, Q > 0$ such that

$$K\rho_{p;(F_2,V_2)}(x,y) \leq \rho_{p;(F_1,V_1)}(x,y) \leq Q\rho_{p;(F_1,V_1)}(x,y)$$

for any $x, y \in \Omega$.

Theorem 4. Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$ and $n-1 < p \leq n$ ($1 \leq p \leq 2$, if $n = 2$). Then the quantity $\rho_{p;(F,V)}(x,y)$ is an intrinsic metric in Ω.

Corollary . Suppose $n-1 < p \leq n$ and $\rho_{p;(F_1,V_1)}$, $\rho_{p;(F_2,V_2)}$ are two p-capacitory metrics on Ω. The metric spaces $\left\{H_\rho, \rho_{p;(F_1,V_1)}\right\}$ and $\left\{H_\rho, \rho_{p;(F_2,V_2)}\right\}$ are quasi-isometric, i.e there exist two constants $K, Q > 0$ such that

$$K\rho_{p;(F_2,V_2)}(x,y) \leq \rho_{p;(F_1,V_1)}(x,y) \leq Q\rho_{p;(F_1,V_1)}(x,y)$$

for any $x, y \in H_p$.

Theorem 5. The topology induced by a p-capacitory metric $\rho_{p;(F,V)}$ into the domain $\Omega \subset \mathbb{R}^n$, $n-1 < p \leq n$ ($1 \leq p \leq 2$ if $n = 2$), coincides with the Euclidean topology.

Definition For arbitrary point $h \in H_\rho$ we consider the balls $B(h, \varepsilon)$, $\varepsilon > 0$, defined in the p-capacitory metric $\rho_{p;(F,V)}$.

Call the set

$$s_h = \bigcap_{\varepsilon>0} \overline{B(h, \varepsilon) \cap \Omega} \subset \overline{\mathbb{R}^n}$$

the support of a boundary element $h \in H_\rho$.

If Ω is locally connected at a point $x \in \partial\Omega$. Then for every sequence $\{x_m \in \Omega\}$ and $h \in H_p$ such that $|x_n - x| \to 0$ we have
$\rho_{p;(F,V)}(x_n, h) \to 0$ while $n \to \infty$ for any $x \in s_h$.

Theorem 6. Let a domain Ω is locally connected at any point $x \in \partial\Omega$. Then the identical mapping $i : \Omega \to \Omega$ can be extend to a homeomorphism $\tilde{i_p} : \tilde{\Omega}_p \to \overline{\Omega}$ if and only if all supports s_h of a boundary elements $h \in H_\rho$ are one-points.

By the classical Lusin theorem every measurable function is uniformly continuous outside of an open set of arbitrary small measure.

It is reasonable to conjecture that every function $u \in L^1_p(\Omega)$ is uniformly continuous outside of an open subset of $\Omega \subset R^n$ of arbitrary small p-capacity.

Unfortunately this conjecture is not correct for an arbitrary domain and is correct only under some additional conditions on Ω.

Definition Let us call a domain Ω a p-Lusin domain if for every function $u \in L_p^1(\Omega)$ and any $\varepsilon > 0$ there exists an open set U_ε of the p-capacity less then ε and such that the function u is uniformly continuous for the p-capacitory metric if it is restricted to the complement of U_ε in Ω.

The unit ball $B(0,1) \subset \mathbb{R}^n$ is an example of a p-Lusin domain.

A domain $\Omega \subset R^n$ is said to be a Sobolev L_p^1 -extension domain if there exists a bounded linear operator $E : L_p^1(\Omega) \to L_p^1(R^n)$ such that for any $u \in L_p^1(\Omega)$ the condition $E(u)|_\Omega = u$ holds.

Theorem 7. If a bounded domain Ω is a L_p^1 -extension domain then the identity mapping $id : H_p \to \bar{\Omega}$ is a homeomorphism for any $n-1 < p \leq n$.

Corollary. Any bounded L_p^1 -extension domain possesses p-Lusin property.

Definition . A domain $\Omega \subset R^n$ is said to be a Sobolev L_p^1 -quasi-extension domain if for any $\varepsilon > 0$ there exist such open set U_ε of p-capacity less then ε that $\Omega \setminus \bar{U_\varepsilon}$ is a L_p^1 -extension domain.

Typical examples of such domains are domains with boundary singularities of p-capacity zero.

Theorem 8. Suppose that Ω is an arbitrary p-Lusin domain in R^n, $n-1 < p \leq n$. For any function $u \in L_p^1(\Omega)$ there exists a function $\tilde{u} : \left(\widetilde{\Omega}_p, \rho_{p;(F,V)}\right) \to R$ defined p-quasi everywhere on H_p such that $\tilde{u}|_{\Omega} = u$.

Theorem 9. Suppose D and Ω are two Euclidean domains and $\varphi : D \to \Omega$ is
a p-quasiconformal homeomorphism, $n-1 < p \leq n$. Then there exists a unique Lipschitz mapping

$$\widetilde{\varphi^{-1}} : \left(\widetilde{\Omega}, \rho_{p;(\varphi(F),\varphi(V))}\right) \to \left(\widetilde{D}, \rho_{p;(F,V)}\right)$$

such that $\widetilde{\varphi^{-1}}|\Omega = \varphi^{-1}$.

EXAMPLES

PEAKS

Definition . Suppose M is a compact $n-1$-dimensional Lipschitz manifold with boundary. Call a warped product the manifold $X := (0, R] \times Int(M)$ with the metric $g := d\rho^2 + f(\rho)\sigma$, where $f(\rho)$ is a positive continuous function and σ is the Riemannian metric of M.

Usually $f(\rho)$ called a warped function of the warped product X.

A domain $\Omega \in R^n$ is called a domain with a peak singularity if it is a bi-Lipschitz domain with an isolated singular point x_o (on its boundary) that has a neighborhood bi-Lipschitz homeomorphic to a warped product $X := (0, R] \times Int(M)$. The warped function $f(\rho)$ represents type of the peak.

RIDGES

Definition . For any $n-m$-dimensional warped product $X := (0, R] \times Int(M)$ (with the metric $g := d\rho^2 + f(\rho)\sigma$) call its product $Y := X \times N$ to an m-dimensional compact manifold as a m-dimensional ridge.

Proposition . For any $0 < r < R$ we have

$$Cap_p(X_r, M_R) = |M| \left(\int_r^R f(\rho)^{\frac{-1}{p-1}} d\rho \right)^{1-p}$$

where $|M|$ is volume of M.

Corollary . $\lim_{r \to 0} Cap_p(X_r, M_R) = 0$ if and only if $\int_0^R f(\rho)^{\frac{-1}{p-1}} d\rho = \infty$.