On Derived Witt Groups

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January 30th, 2013
Thematic Program Postdoctoral Seminar

Outline

Symmetric Forms

Let k be a field.
A *symmetric bilinear form over* k is a pair $(V, \mathfrak{b})$ consisting of:

- V a finite dimensional vector space over k;
- $\mathfrak{b} : V \times V \to k$ symmetric and bilinear.

A symmetric bilinear form is *non-degenerate* if $l_{\mathfrak{b}} \in \mathrm{Hom}_k(V, V^*)$ is an isomorphism, where $l_{\mathfrak{b}}$ is given by the assignment

$$v \mapsto \mathfrak{b}(v, -)$$

Witt Group of a Field: Origin

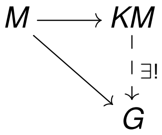
- Ernst Witt:
  Theorie der quadratischen Formen in beliebigen Körpern,
  J. reine angew. Math. 176 (1937) 31-44.

Grothendieck Group of an Abelian Monoid: Definition

Let M be an abelian monoid, that is, $(M, +)$ is a set M with an operation $+$ satisfying, with the exception of inverses, the axioms of an abelian group.

The *Grothendieck group of M* is an abelian group KM equipped with a morphism of monoids $M \to KM$ satisfying the following universal property:

for any abelian group G and morphism of monoids $M \to G$



The Witt Group of a Field: First Step

Let k be a field. Let $\mathrm{Sym}(k)$ denote the abelian monoid $(\mathrm{Sym}, \perp)$ of isometry classes of non-degenerate symmetric bilinear forms over k equipped with the operation of orthogonal sum $\perp$.

Let $\mathrm{KSym}(k)$ denote its Grothendieck group:

The elements of $\mathrm{KSym}(k)$ are formal differences $[\mathfrak{b}_1] - [\mathfrak{b}_2]$ of classes with the rule that

$$[\mathfrak{b}_1] - [\mathfrak{b}_2] = [\mathfrak{b}_1^{'}] - [\mathfrak{b}_2^{'}] \Leftrightarrow \mathfrak{b}_1 \perp \mathfrak{b}_2^{'} \perp \mathfrak{b} \simeq \mathfrak{b}_1^{'} \perp \mathfrak{b}_2 \perp \mathfrak{b}$$

The map $\mathrm{Sym}(k) \to \mathrm{KSym}(k)$ sends $[\mathfrak{b}]$ to $[\mathfrak{b}] - [0]$.

The Witt Group of a Field

Let k be a field. Let $\mathbb{H}$ denote the hyperbolic form and let $\langle \mathbb{H} \rangle \subset \mathrm{KSym}(k)$ denote the subgroup generated by $[\mathbb{H}] - [0]$. The quotient

$$\mathrm{KSym}(k)/\langle \mathbb{H} \rangle$$

is called the *Witt group of k* and denoted by $W(k)$.

Metabolic Forms

Let $(V, \mathfrak{b})$ be a symmetric bilinear form.
A subspace $i : L \hookrightarrow V$ is said to be a *Lagrangian of V* if $V \xrightarrow{l_{\mathfrak{b}}} V^* \xrightarrow{i^*} L^*$ is surjective and $L = L^{\perp} := \ker(V \to L^*)$, that is, the sequence below is exact

$$0 \to L \to V \to L^* \to 0$$

$(V, \mathfrak{b})$ *metabolic* $\overset{\text{dfn}}{\Leftrightarrow}$ $(V, \mathfrak{b})$ has a Lagrangian $L \subset V$.

The Witt Group Revisited

- (Fact) If $(V, \mathfrak{b})$ is metabolic with Lagrangian $L \subset V$, then in $\mathrm{KSym}(k)$

$$[\mathfrak{b}] = [\mathbb{H}(L)]$$

- Let $\mathfrak{M}$ denote the subgroup of $\mathrm{KSym}(k)$ generated by metabolic forms. The inclusion $\langle \mathbb{H} \rangle \subset \mathfrak{M}$ induces an isomorphism of quotients

$$\mathrm{KSym}(k)/\langle \mathbb{H} \rangle \xrightarrow{\simeq} \mathrm{KSym}(k)/\mathfrak{M}$$

Ringed Spaces

Let $(X, \mathcal{O}_X)$ be a *ringed space*:

- A topological space X;
- A *sheaf of rings* $\mathcal{O}_X$ on X:

(i) For every open U in X, a commutative ring $\mathcal{O}_X(U)$ with unit;

(ii) For every open V in U, a morphism of rings

$$\rho_V^U : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$$

such that for any open subset U, $\rho_U^U = \mathrm{Id}_{\mathcal{O}_X(U)}$;
If we have three open subsets $W \subset V \subset U$, then
$\rho_W^U = \rho_W^V \circ \rho_V^U$;

(iii) $\mathcal{O}_X$ must satisfy sheaf axiom.

Examples of Ringed Spaces

Example 1

Let k be a field. We will construct a ringed space $(X, \mathcal{O}_X)$ that will be denoted by $\operatorname{Spec} k$

- (Topological space)
  Let $X := \{*\}$ be the topological space consisting of a point.
- (Sheaf of rings)
  Define $\mathcal{O}_X(X) := k$, $\mathcal{O}_X(\varnothing) := 0$, and

  $$\rho^X_\varnothing : k \to 0$$

  to be the zero morphism.

Examples of Ringed Spaces

Example 2

Let X be a scheme.

A scheme is, by definition, a ringed space $(X, \mathcal{O}_X)$ satisfying additional hypotheses:
‘locally the spectrum of a commutative ring with unit’.

$\mathcal{O}_X$-modules

Let $(X, \mathcal{O}_X)$ be a ringed space. An *$\mathcal{O}_X$-module* $\mathcal{E}$ consists of:

(i) For every open U in X, an $\mathcal{O}_X(U)$-module $\mathcal{E}(U)$;

(ii) For every open V in U, a homomorphism of abelian groups $\mathcal{E}(U) \to \mathcal{E}(V)$ satisfying:

for every $a \in \mathcal{O}_X(U)$, $e \in \mathcal{E}(U)$,

$$(af)|_V = a|_V \, f|_V$$

(iii) $\mathcal{E}$ must satisfy sheaf axiom.

Vector Bundles

Let X be scheme. An $\mathcal{O}_X$-module $\mathcal{E}$ is:

- *free of rank n* if $\mathcal{E} \backsimeq \mathcal{O}_X^n$;
- *locally free of rank n* if there exists an open covering X_i of X such that $\mathcal{E}|_{X_i}$ is free of rank n for every i;
- *locally free of finite rank* if there exists an open covering X_i of X and integers n_i such that $\mathcal{E}|_{X_i}$ is locally free of rank n_i. In this case we call $\mathcal{E}$ a *vector bundle on X*.

A Property of Vector Bundles

Fact

Let X be a scheme and $\mathcal{E}$ a vector bundle on X. Then:

- the $\mathcal{O}_X$-module $\mathcal{E}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is a vector bundle;
- there is an isomorphism of vector bundles

$$can : \mathcal{E} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X), \mathcal{O}_X)$$

that is, $\mathcal{E} \simeq \mathcal{E}^{**}$.

Symmetric Bilinear Forms over a Scheme

Let X be a scheme. A *symmetric bilinear form* $(\mathcal{E}, \mathfrak{b})$ on X is:

- a vector bundle $\mathcal{E}$ on X, *i.e.,* , an $\mathcal{O}_X$-module locally free of finite rank;
- a morphism $\mathfrak{b} : \mathcal{E} \times \mathcal{E} \to \mathcal{O}_X$ of sheaves equipped with, for every open U in X, a symmetric bilinear form over $\mathcal{O}_X(U)$

$$\mathfrak{b} : \mathcal{E}(U) \times \mathcal{E}(U) \to \mathcal{O}_X(U)$$

Metabolic Forms

Let $(\mathcal{E}, \mathfrak{b})$ be a symmetric bilinear form on a scheme X.

- A *Lagrangian of* $\mathcal{E}$ is a vector bundle $\mathcal{L} \hookrightarrow \mathcal{E}$ which sits in an exact sequence of vector bundles

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^* \to 0$$

- $(\mathcal{E}, \mathfrak{b})$ is *metabolic* if it has a Lagrangian.

Witt Group of a Scheme

- Manfred Knebusch's habilitation thesis: *Grothendieck-Und Wittringe Von Nichtausgearteten Symmetrischen Bilinearformen*, S.-Ber. Heidelberg. Acad. Wiss. Math. 3. Abh.(1970).
- The Witt group of a scheme is a globalization to schemes of the Witt group of a field, that is,

$$W(\mathrm{Spec}\ k) = W(k).$$

Definition of the Witt Group of a Scheme

Let X be a scheme.

- Let $\mathrm{KSym}(X)$ denote the Grothendieck group of the abelian monoid $\mathrm{Sym}(X)$.
- Let $\mathfrak{M} \subset \mathrm{KSym}(X)$ denote the subgroup generated by metabolic forms $[\mathcal{M}] - [0]$.
- The abelian group obtained by taking the quotient

$$\mathrm{KSym}(k)/\mathfrak{M}$$

is called the *Witt group of X* and denoted by $W(X)$.

An Example

Theorem (Arason, J.(1980))

Let k be a field of characteristic not 2 and let $n \geq 1$. Then the structure morphism $\mathbb{P}^n_k \to \operatorname{Spec} k$ induces an isomorphism of Witt groups

$$W(k) \xrightarrow{\cong} W(\mathbb{P}^n_k)$$

Balmer's Witt Group of a Triangulated Category

- Paul Balmer's thesis:
  c.f. *Derived Witt groups of a scheme*, Journal of Pure and Applied Algebra 141, no 2 (1999).
- Knebusch's Witt group $W(X)$ of a scheme becomes a cohomology theory $W^n(X)$ with $W^0(X) = W(X)$.

Symmetric Forms in a Triangulated Category with Duality

Let $(\mathcal{A}, \sharp)$ be a small triangulated category with duality (TriCatD).

- A *non-degenerate symmetric form* in $(\mathcal{A}, \sharp)$ is a pair consisting of an object X of $\mathcal{A}$ and an isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{A}}(X, X^\sharp)$ such that $\varphi = \varphi^t$.
- Let $\mathrm{Sym}((\mathcal{A}, \sharp))$ denote the abelian monoid of isometry classes of non-degenerate symmetric forms with respect to orthogonal sum $\perp$.

Metabolic Forms

Let (X, φ) be a non-degenerate symmetric form in $(\mathcal{A}, \sharp)$.

- A *Lagrangian of X* is an object L which sits in a distinguished triangle

$$L \xrightarrow{i} X \xrightarrow{i^\sharp \varphi} L^\sharp \xrightarrow{\psi} T(L)$$

  such that ψ satisfies a symmetry condition.
- (X, φ) is *metabolic* if it has a Lagrangian L.

The Witt Group of a Triangulated Category with Duality

Let $(\mathcal{A}, \sharp)$ be a TriCatD.

- Let $\mathrm{KSym}((\mathcal{A}, \sharp))$ denote the Grothendieck group of the abelian monoid $\mathrm{Sym}((\mathcal{A}, \sharp))$.
- Let $\mathfrak{M} \subset \mathrm{KSym}(X)$ denote the subgroup generated by metabolic forms $[\mathcal{M}] - [0]$.
- The abelian group obtained by taking the quotient

$$\mathrm{KSym}((\mathcal{A}, \sharp))/\mathfrak{M}$$

is called the *Witt group of* $(\mathcal{A}, \sharp)$ and denoted by $W((\mathcal{A}, \sharp))$.

Definition of the Witt Group W^n

- For $n \in \mathbb{Z}$, shifting the duality, produces a TriCatD $(\mathcal{A}, \sharp^{[n]})$.
- The *triangulated Witt group* of $(\mathcal{A}, \sharp^{[n]})$ is denoted

$$W^n((\mathcal{A}, \sharp))$$

Some Properties of the Triangulated Witt Group

- 4-periodicity: For $n \in \mathbb{Z}$, $W^n((\mathcal{A}, \sharp)) \simeq W^{n+4}((\mathcal{A}, \sharp))$.
- Localization: "Short exact sequence" of TriCatD with 2 invertible $\mathcal{J} \xrightarrow{i} \mathcal{A} \xrightarrow{j} S^{-1}\mathcal{A}$ induces long exact sequence

$$\cdots \to W^n(\mathcal{J}) \xrightarrow{i} W^n(\mathcal{A}) \xrightarrow{j} W^n(S^{-1}\mathcal{A}) \xrightarrow{\partial} W^{n+1}(\mathcal{J}) \to \cdots$$

Derived Witt Groups

Let X be a scheme with $\frac{1}{2} \in \mathcal{O}_X(X)$.

- Let $\text{Vect}(X)$ denote the category of locally free of finite rank $\mathcal{O}_X$-modules, and $D^b(\text{Vect}(X))$ its bounded derived category.
- The dual $\mathcal{E}^*$ induces a duality $\sharp = *$ on the derived category making $(D^b(\text{Vect}(X)), *)$ a TriCatD having 2 invertible.
- For $n \in \mathbb{Z}$, the *derived Witt groups* of X, denoted $W^n(X)$, are the triangulated Witt groups $W^n((D^b(\text{Vect}(X)), *))$.

Some Properties of the Derived Witt Groups

- Agreement with Knebusch: $W^0(X) = W(X)$.
- Localization: For Z closed in X with open complement U, there is a long exact sequence

$$\cdots \to W_Z^n(X) \to W^n(X) \to W^n(U) \to W_Z^{n+1}(X) \to \cdots$$

- Homotopy invariance: $W^n(X \times_k \mathbb{A}_k^1) \stackrel{\simeq}{\leftarrow} W^n(X)$.

An Example

Theorem (Walter, C. (2003) c.f. Nenashev, A.)

Let X be a regular scheme with $1/2 \in \mathcal{O}_X(X)$ and $r \geq 1$. Let $\mathbb{P}^r_X$ be the projective space over X. For r even,

$$W^i(\mathbb{P}^r_X) = W^i(X)$$

For r odd,

$$W^i(\mathbb{P}^r_X) = W^i(X) \oplus W^{i-r}(X)$$

- Derived Witt and Grothendieck-Witt groups follow a development very similar to algebraic K-theory.
- Algebraic K-theory has been studied considerably more.
- We prove Witt and Grothendieck-Witt analogues of some important theorems for algebraic K-theory.

Finite Generation Question

Let X be a smooth variety over a finite field of characteristic $\neq 2$. Are the Witt groups $W^n(X)$ finitely generated as abelian groups?

Motivation

Known result:
If the Witt groups $W^n(X)$ and the algebraic K-groups $K_n(X)$ are finitely generated abelian groups, then the Grothendieck-Witt groups $GW^n_m(X)$ are finitely generated.

Situation in K-theory

Let X be a smooth variety over a finite field.

- Quillen (1974):
  If $\dim X = 1$, then the algebraic K-groups $K_n(X)$ are finitely generated as abelian groups.
- Known implication:
  If the motivic cohomology groups $H^p_{mot}(X, \mathbb{Z}(q))$ are finitely generated, then the algebraic K-groups $K_n(X)$ are finitely generated.

Witt Groups

- Known result:
  - The $W^n(X)$ are torsion abelian groups.
  - So, $W^n(X)$ finitely generated if and only if $W^n(X)$ finite.
- Arason, Elman, and Jacob (1991):
  - If X is a complete regular curve over a finite field, then $W(X)$ is finite.
  - That is, they proved $W^0(X)$ is finite.

Publication

Jeremy Jacobson:
Finiteness theorems for the shifted Witt and higher Grothendieck-Witt groups of arithmetic schemes, Journal of Algebra, Volume 351, Issue 1, 1 February 2012, Pages 254 - 278.

Absolute Finiteness

Theorem (Absolute Finiteness, Theorem 3.33)

Let X be a smooth variety over a finite field of characteristic $\neq 2$ with $\dim X \leq 2$. Then, the Witt groups $W^n(X)$ are finite for all $n \in \mathbb{Z}$.

Relative Finiteness

Theorem (Relative Finiteness, Theorem 3.36)

Let X be a connected variety that is proper and smooth over a finite field of characteristic $\neq 2$. If $\dim X = 3$, then:

(i) *The Witt groups $W^1(X)$ and $W^3(X)$ are finite;*

(ii) *Finiteness of $W^0(X)$ is equivalent to finiteness of $W^2(X)$;*

(iii) *$W^0(X)$ is finite if and only if the motivic cohomology groups $H^p_{mot}(X, \mathbb{Z}/2\mathbb{Z}(q))$ are finite for all $p, q \in \mathbb{Z}$.*

Relative Finiteness Continued

Theorem (Relative Finiteness: Continued)

Let X be a smooth variety over a finite field of characteristic $\neq 2$. Then:

(*i*) *(Theorem 3.34) If the motivic cohomology groups $H^p_{mot}(X, \mathbb{Z}/2\mathbb{Z}(q))$ are finite for all $p, q \in \mathbb{Z}$, then the Witt groups $W^n(X)$ are finite for all $n \in \mathbb{Z}$;*

(*ii*) *(Theorem 5.4) If the motivic cohomology groups $H^p_{mot}(X, \mathbb{Z}(q))$ are finitely generated for all $p, q \in \mathbb{Z}$, then the Grothendieck-Witt groups $GW^m_n(X)$ are finitely generated for all $m, n \in \mathbb{Z}$.*

Concluding Remark on the Finite Generation Question

- Finite generation of $H^p_{mot}(X, \mathbb{Z}(q))$ (resp. $H^p_{mot}(X, \mathbb{Z}/2\mathbb{Z}(q))$) for all $p, q \in \mathbb{Z}$ is only known when $\dim X \leq 1$ (resp. $\dim X \leq 2$).

Gersten Conjecture

Let A be a regular local ring with 2 invertible and with fraction field $\mathrm{Frac}\, A = K$, and $X = \mathrm{Spec}\, A$.
Is the *Gersten complex for the Witt groups*

$$0 \to W(X) \to W(K) \to \bigoplus_{x \in X^1} W(k(x)) \to \cdots \to \bigoplus_{x \in X^d} W(k(x)) \to 0$$

an exact complex?

Motivation

- Known cases of the Gersten conjecture are used in the proofs of many theorems.

Situation in K-theory

- Let A be a regular local ring with fraction field $\mathrm{Frac}\, A = K$, and $X = \mathrm{Spec}\, A$.
  The *Gersten complex for K-theory* is, for $n \geq 0$,
  $$0 \to K_n(X) \to K_n(K) \to \cdots \to \bigoplus_{x \in X^d} K_{n-d}(k(x)) \to 0.$$
- In 1972, Gersten conjectured that the Gersten complex is exact for A a regular local ring.

Gersten Conjecture for K-theory

- Quillen (1972) proved case that A is essentially smooth over a field.
- Bloch (1983) proved the Gersten complex is exact for $A[\pi^{-1}]$, where A is a local ring essentially smooth over a discrete valuation ring (DVR) Λ with uniformizing parameter π.
- Gillet & Levine (1987) proved, for K-theory with finite coefficients, the case that A is a local ring essentially smooth over a DVR.
- Panin (2001) proved case that A is a regular local ring containing a field.

Gersten Conjecture for Witt Groups

- Balmer (2001) proved case that A is essentially smooth over a field.
- Balmer, Gille, Panin, Walter (2001) proved case that A is a regular local ring containing a field.
- Gille & Hornbostel (2006) proved Gersten conjecture in case that A is essentially smooth over a field using Quillen normalization.

Exactness of Gersten Complex of Generic Fiber

Theorem (Witt Analogue of Bloch's Theorem, Theorem 4.19)

Let Λ be a DVR with uniformizing parameter π and residue field of characteristic $\neq 2$, and let A be a local ring essentially smooth over Λ. Then, the Gersten complex for the Witt groups of $A[\pi^{-1}]$ is exact.

Essentially Smooth over DVR Case

Corollary (Corollary 4.20)

Let Λ be a DVR with residue field of characteristic $\neq 2$, and let A be a local ring essentially smooth over Λ. Then, the Gersten conjecture is true for A, that is, the Gersten complex is exact for the Witt groups of A.

Regular over DVR Case

Theorem (Theorem 4.29)

Let Λ be a DVR with residue field of characteristic $\neq 2$, and let A be a local ring regular over Λ. Then, the Gersten conjecture is true for A, that is, the Gersten complex is exact for the Witt groups of A.

The End

Thank you for listening.