The Structure of Quantum Line Bundles over Quantum Teardrops

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Quantum vector bundle

- Quantum $\equiv$ Noncommutative
- (Compact) "quantum space" $X_q \longleftrightarrow$ (Unital) noncommutative C*-algebra $C(X_q)$
  - Often a dense *-subalgebra $\mathcal{O}(X_q)$ of $C(X_q)$ is used in place of $C(X_q)$ to avoid unnecessary technical inconveniences in early stage of development

- Swan's Theorem suggests:

Isomorphism classes of "Quantum vector bundle" E_q over a compact quantum space X_q

$\longleftrightarrow$ Isomorphism classes of finitely generated *left* projective module $\Gamma(E_q)$ over $C(X_q)$

$\longleftrightarrow$ Unitary equivalence classes of projections in $M_\infty(C(X_q))$

Examples of quantum vector bundles

- Example: For $l \in \mathbb{N}$ and $\mathcal{K}$ the algebra of all compact operators, the projections $\oplus_{j=1}^{l} p_{k_j} \in M_1\left(\left(\mathcal{K}^l\right)^+\right)$ with $k_j \geq 0$ and $I_{r-1} \oplus \left(I - \left(\oplus_{j=1}^{l} p_{n_j}\right)\right) \oplus \left(\oplus_{j=1}^{l} p_{m_j}\right) \in M_{r+1}\left(\left(\mathcal{K}^l\right)^+\right)$ with $r \in \mathbb{N}$, $n_j, m_j \geq 0$ such that $n_j m_j = 0$ represent all unitarily inequivalent classes of projections in $M_\infty\left(\left(\mathcal{K}^l\right)^+\right)$ where $p_k := \sum_{i=1}^{k} e_{ii} \in \mathcal{K}$ and

$$0 \to \mathcal{K}^l \equiv \oplus_{j=1}^{l} \mathcal{K} \to \left(\mathcal{K}^l\right)^+ \equiv C(W\mathbb{P}_q(k,l)) \to \mathbb{C} \to 0 \;\; \text{exact}$$

- Example: (K. Bach) The projections $1 \otimes p_k$ with $k \geq 0$ and I_r with $r \in \mathbb{N}$ represent all unitarily inequivalent classes of projections in $M_\infty\left(C\left(S_q^3\right)\right)$ where

$$0 \to C(\mathbb{T}) \otimes \mathcal{K} \to C\left(SU(2)_q\right) \equiv C\left(S_q^3\right) \to C(\mathbb{T}) \to 0 \;\; \text{exact}$$

Quantum group

• (Woronowicz, Van Daele, ...) A compact quantum group is a unital separable C*-algebra A with comultiplication Δ such that $(A \otimes 1)\,\Delta A$ and $(1 \otimes A)\,\Delta A$ are dense in A.

- (Woronowicz) Compact quantum group A contains a dense *-subalgebra $\mathcal{A}$, forming a Hopf *-algebras $(\mathcal{A}, \Delta, ^*, S, \varepsilon)$, and has a Haar state $h \in A^*$ satisfying $h(1) = 1$ and

$$(\mathrm{id} \otimes h)\,\Delta a = h(a)\,1 = (h \otimes \mathrm{id})\,\Delta a.$$

Δ, ε: $\mathbb{C}$-linear *-algebra homomorphism
S: $\mathbb{C}$-linear algebra anti-automorphism
$S\left(S\left(\cdot\right)^*\right)^* = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\,\Delta = (\varepsilon \otimes \mathrm{id})\,\Delta$
$\mu\,(\mathrm{id} \otimes S)\,\Delta = \mu\,(S \otimes \mathrm{id})\,\Delta = \varepsilon$

- We denote $\mathcal{A}$ by $\mathcal{O}(G_q)$ if A is denoted as $C(G_q)$.

Quantum homogeneous space

• For a quantum subgroup H_q of a compact quantum group G_q given by a surjective Hopf algebra homomorphism $\mathcal{O}(G_q) \to \mathcal{O}(H_q)$, the *-subalgebra

$$\mathcal{O}(G_q/H_q) := \{x \in \mathcal{O}(G_q) : \Delta_R(x) = x \otimes 1\}$$

of *coinvariants* of the coaction $\mathcal{O}(G_q) \overset{\Delta_R}{\to} \mathcal{O}(G_q) \otimes \mathcal{O}(H_q)$ defines a "quantum homogeneous space" G_q/H_q.
• Example: $S_q^{2n+1} = SU(n+1)_q / SU(n)_q$ with $q \in (0,1)$ generated by $z_0, ..., z_n$ subject to $\sum_{m=0}^{n} z_m z_m^* = 1$, $z_i z_j = q z_j z_i$ for $i < j$, $z_i z_j^* = q z_j^* z_i$ for $i \neq j$, and $z_i z_i^* = z_i^* z_i + \left(q^{-2} - 1\right) \sum_{m=i+1}^{n} z_m z_m^*$.

Quantum quotient space

• More generally, given a coaction $\Delta_R : \mathcal{O}(X_q) \to \mathcal{O}(X_q) \otimes \mathcal{O}(H_q)$ of a compact quantum group H_q on a compact quantum space X_q, the *-subalgebra

$$\mathcal{O}(X_q/H_q) := \{x \in \mathcal{O}(X_q) : \Delta_R(x) = x \otimes 1\}$$

of *coinvariants* defines a "quantum quotient space" X_q/H_q.
• Example: The quantum weighted complex projective space $WP_q(l_0, ..., l_n)$, for pairwise coprime numbers $l_0, ..., l_n \in \mathbb{N}$, is defined as the quantum quotient space for the coaction of $\mathcal{O}\left(U(1)_q\right) \equiv \mathcal{O}(U(1)) = \mathbb{C}[u, u^*]$ on $\mathcal{O}\left(S_q^{2n+1}\right)$ defined by

$$z_i \in \mathcal{O}\left(S_q^{2n+1}\right) \mapsto z_i \otimes u^{l_i} \in \mathcal{O}\left(S_q^{2n+1}\right) \otimes \mathcal{O}(U(1)) \quad \text{for } i = 0, ..., n$$

- When $l_0 = ... = l_n = 1$, we get the quantum complex projective $\mathbb{CP}_q^n$.
- For $n = 1$, we get the so-called quantum teardrop $WP_q(k, l)$ with k, l coprime.

Quantum principal bundle

- Brzeziński and Fairfax determined that the algebra $\mathcal{O}\left(S_q^3\right)$ is a principal $\mathcal{O}\left(U\left(1\right)\right)$-comodule algebra over $\mathcal{O}\left(WP_q\left(k,l\right)\right)$ if and only if $k=l=1$.
  - Consistent with the classical $U\left(1\right)$-action $\left(z,w\right)\mapsto\left(u^k z, u^l w\right)$ (with k,l coprime) for $u\in\mathbb{T}$ on $\mathbb{S}^3$.

- They found that the quantum lens space $L_q\left(l;1,l\right)$ provides the total space of a quantum $U\left(1\right)$-principal bundle over $WP_q\left(1,l\right)$, where $L_q\left(l;1,l\right)$ is the quantum quotient space defined by the coaction $\rho:\mathcal{O}\left(S_q^3\right)\to\mathcal{O}\left(S_q^3\right)\otimes\mathcal{O}\left(\mathbb{Z}_l\right)$ with $\rho\left(\alpha\right)=\alpha\otimes w$ and $\rho\left(\beta\right)=\beta\otimes 1$ where $\alpha:=z_0$ and $\beta:=z_1^*$ generate $\mathcal{O}\left(S_q^3\right)\equiv\mathcal{O}\left(SU\left(2\right)_q\right)$, and w is the unitary group-like generator of $\mathcal{O}\left(\mathbb{Z}_l\right)$ with $w^l=1$.
  - With $\mathcal{O}\left(L_q\left(l;1,l\right)\right)$ generated by $c:=\alpha^l$ and $d:=\beta$, the coaction of $\mathcal{O}\left(U\left(1\right)\right)$ on the quantum $U\left(1\right)$-principal bundle is given by $\rho_l: c\mapsto c\otimes u$ and $\rho_l: d\mapsto d\otimes u^*$.

Quantum line bundle

- The irreducible corepresentations of $\mathcal{O}\left(U\left(1\right)\right)$ on left comodules W_n correspond to exactly the irreducible (1-dimensional) representations of $U\left(1\right)$ indexed by $n \in \mathbb{Z}$.
- Brzeziński and Fairfax found that the cotensor product of $\mathcal{O}\left(L_q\left(l;1,l\right)\right)$ with W_n over $\mathcal{O}\left(U\left(1\right)\right)$ turns out to be a finitely generated projective module $\mathcal{L}\left[n\right]$ over $\mathcal{O}\left(WP_q\left(1,l\right)\right)$ and is naturally called a quantum line bundle over $WP_q\left(1,l\right)$.
  - Following a general procedure, one can compute an idempotent matrix $E\left[n\right]$ over $\mathcal{O}\left(WP_q\left(1,l\right)\right)$ implementing the projective module $\mathcal{L}\left[n\right]$ with complicated entries $E\left[n\right]_{ij} = \omega\left(u^n\right)^{[2]_i}\omega\left(u^n\right)^{[1]_j}$ where $\omega\left(u^n\right) = \sum_i \omega\left(u^n\right)^{[1]_i} \otimes \omega\left(u^n\right)^{[2]_i}$ comes from a *strong connection* $\omega : \mathcal{O}\left(U\left(1\right)\right) \to \mathcal{O}\left(L_q\left(l;1,l\right)\right) \otimes \mathcal{O}\left(L_q\left(l;1,l\right)\right)$.
  - They showed in particular that the $\mathcal{O}\left(WP_q\left(1,l\right)\right)$-module $\mathcal{L}\left[1\right]$ is not free.

Classification

• Brzeziński and Fairfax also determined the structure of the C*-algebra $C\left(WP_q\left(1,l\right)\right)$ as $\left(\mathcal{K}^l\right)^+$ and computed its K-groups.
• It is of interest to identify explicitly the quantum line bundles $\mathcal{L}\left[n\right]$ for all $n \in \mathbb{Z}$ among all finitely generated projective modules over $\left(\mathcal{K}^l\right)^+$ already classified above.

- It turns out that $\mathcal{L}\left[n\right]$ is isomorphic to the projective module represented by projections
  - $I_1 \oplus \left(\oplus_{j=1}^l p_n\right) \in M_2\left(\left(\mathcal{K}^l\right)^+\right)$ if $n \geq 0$
  - $I - \left(\oplus_{j=1}^l p_{-n}\right) \in M_1\left(\left(\mathcal{K}^l\right)^+\right)$ if $n < 0$
- We note that $\mathcal{L}\left[n\right] \otimes_{C\left(WP_q\left(1,l\right)\right)} C\left(S_q^3\right)$ for all $n \in \mathbb{Z}$ is the same rank-1 free module over $C\left(S_q^3\right)$, showing that these non-isomorphic quantum line bundles $\mathcal{L}\left[n\right]$ over $WP_q\left(1,l\right)$ pull back to the same quantum line bundles over S_q^3 via the quotient $S_q^3 \to WP_q\left(1,l\right)$.