# *Rigorous computations for infinite dimensional dynamical systems*

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UNIVERSITÉ LAVAL

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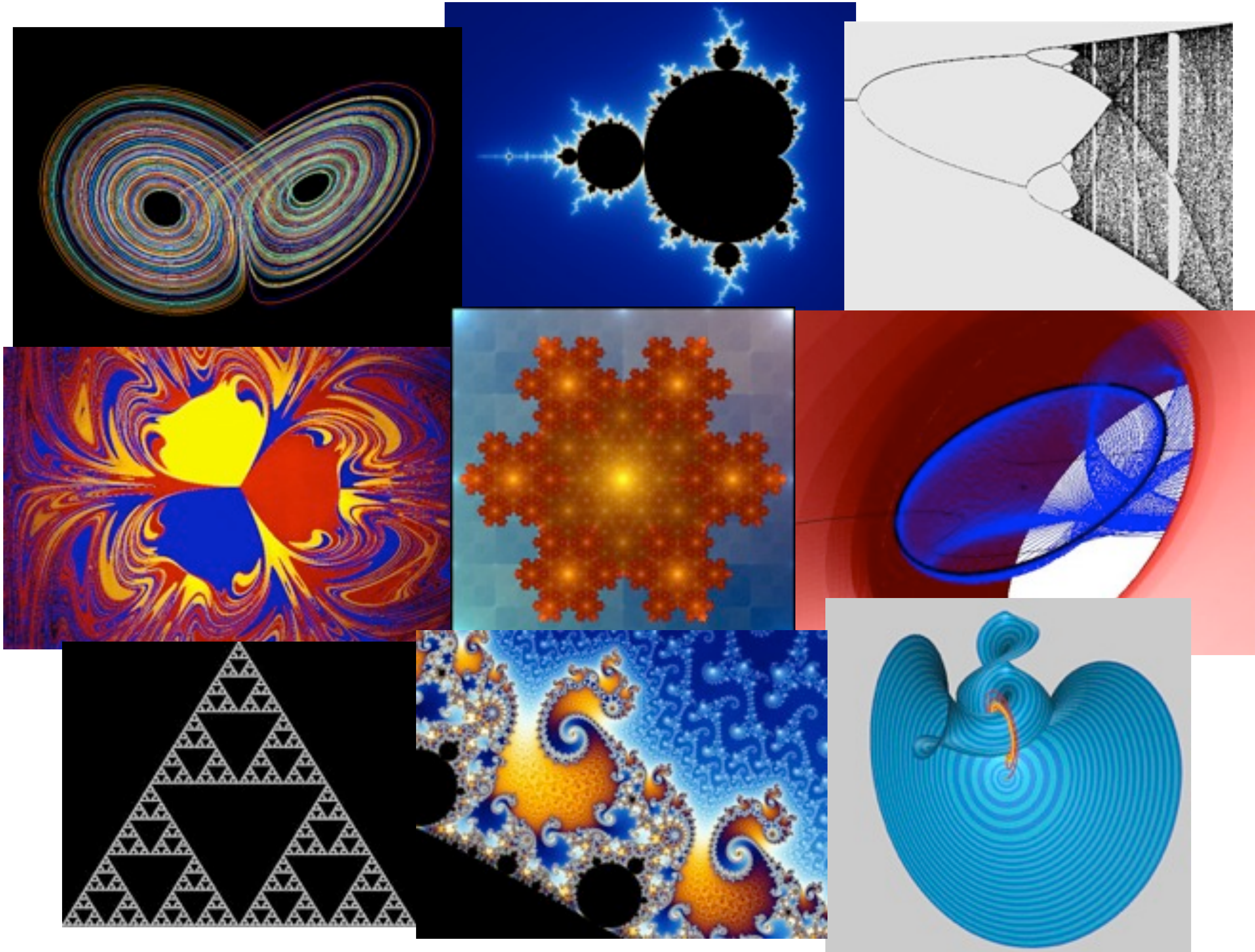

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*The Free Encyclopedia*

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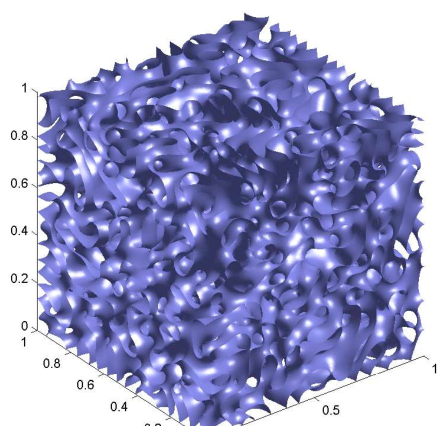
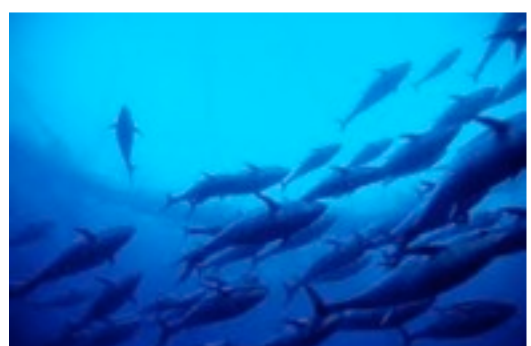
**finance**

**fluids**

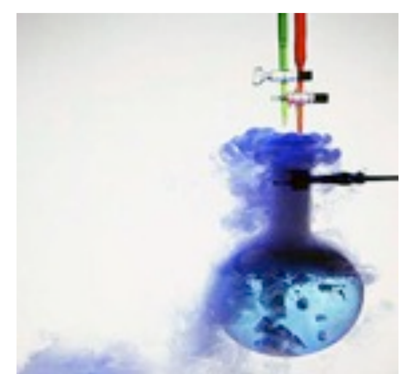
**weather prediction**



**material science**



**population dynamics**



**chemical reactions**

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A dynamical system is a tuple $(T, M, \Phi)$

$T$ : monoid (time)

$M$ : set (state space)

$\Phi$ : map (evolution function)

$$\Phi : T \times M \to M$$

satisfying the two following properties

$$\begin{cases} \Phi(0, x) = x \\ \Phi(t_2, \Phi(t_1, x)) = \Phi(t_1 + t_2, x) \end{cases}$$

$$\forall\ x \in M \text{ and } \forall\ t_1, t_2 \in T$$

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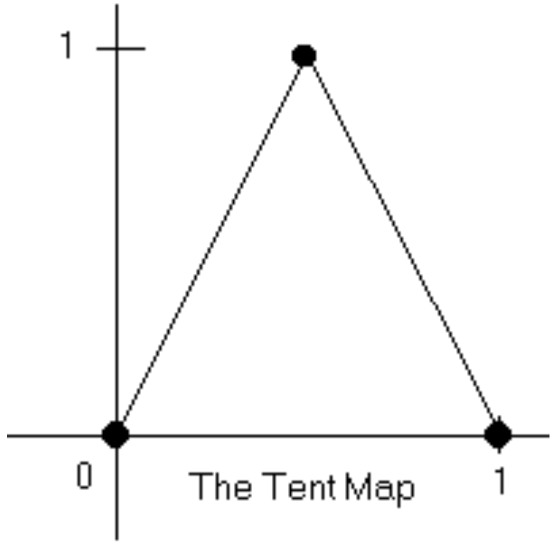
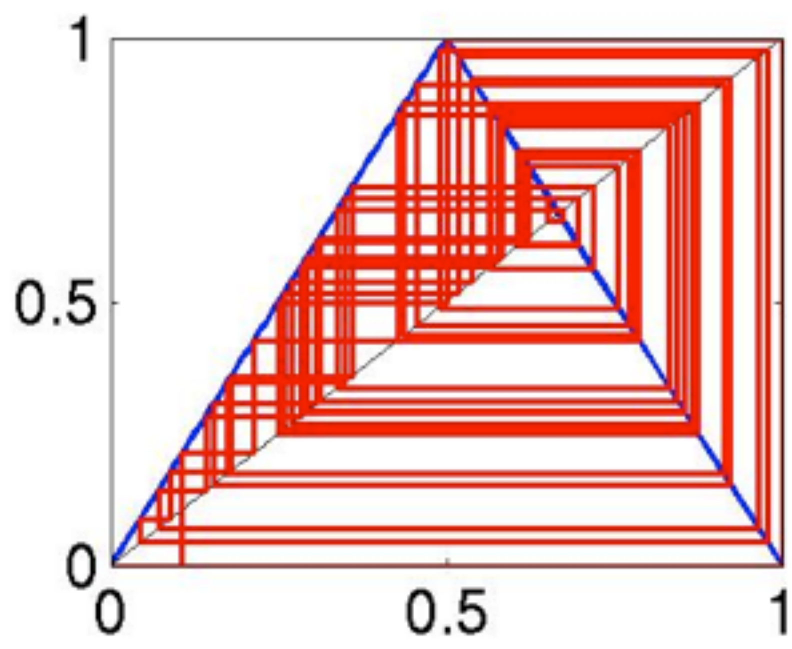
**In case the state space M is a function space, we have an <u>infinite</u> dimensional dynamical system !**

# Examples

## I. Finite dimensional discrete dynamical systems




$$f(x) = \begin{cases} 2x, & \text{for } x \in [0, \frac{1}{2}) \\ 2(1-x), & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$




$T = \mathbb{N}$ (discrete time)

$M = [0, 1]$ (state space)

$$\Phi : T \times M \to M$$
$$(n, x) \mapsto \Phi(n, x) = f^n(x)$$

# Examples

## 2. Finite dimensional continuous dynamical systems: ODEs

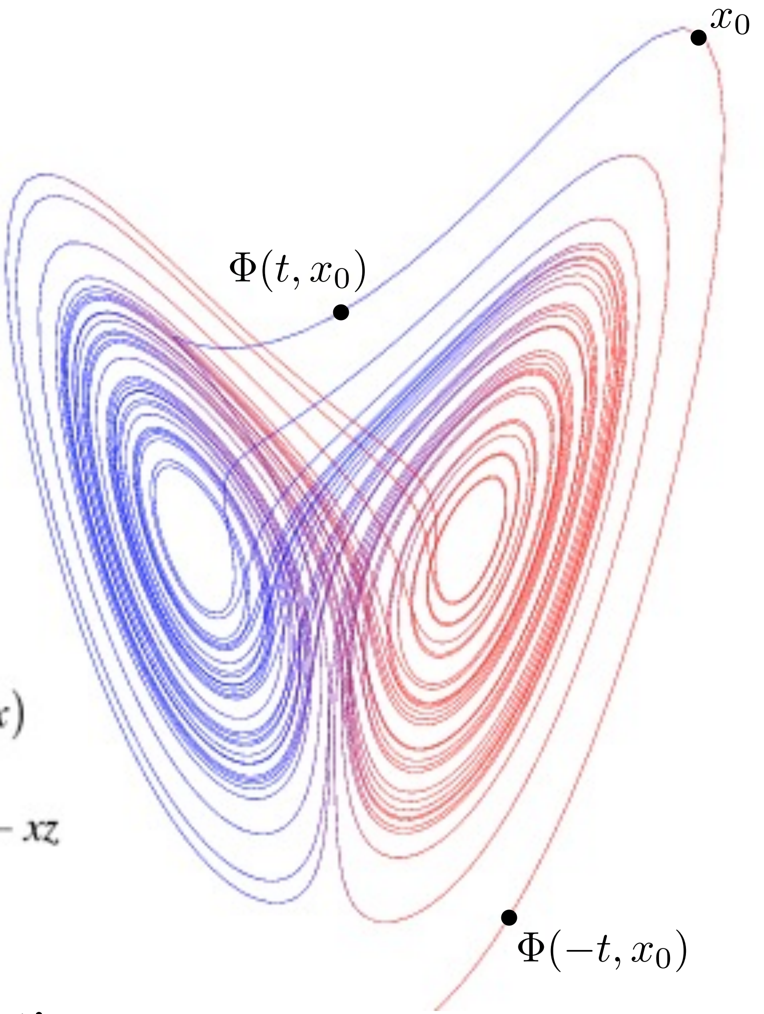
**(IVP)** $\begin{cases} \dfrac{dx}{dt} = f(x) \\ x(0) = x_0 \end{cases}$ $\quad [f \in C^1(\mathbb{R}^n)]$

$\Phi(t, x_0)$ : solution of the (IVP)

$T = \mathbb{R}$ (continuous time)

$M = \mathbb{R}^n$ (state space)

$$\Phi : T \times M \to M$$
$$(t, x_0) \mapsto \Phi(t, x_0)$$




$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = xy - bz$$

**Lorenz equations**

# Examples

## 3. Infinite dimensional continuous dynamical systems

### (a) Partial differential equations

**Cahn-Hilliard equation**

$$\frac{\partial u}{\partial t} - \Delta\left(-\nu \Delta u - u + u^3\right) = 0$$

$$\Omega \subset \mathbb{R}^n,\ n = 1, 2, 3$$

$T = [0, \infty)$ (continuous time)

$M = L^2(\Omega)$ (infinite dimensional state space)

$$\Phi : T \times M \to M$$
$$(t, u_0) \mapsto \Phi(t, u_0)$$ (semigroup)

# Examples

## 3. Infinite dimensional continuous dynamical systems

**(b) Delay differential equations** $y'(t) = \mathcal{F}(y(t), y(t-\tau))$

$T = [0, \infty)$ (continuous time)

$M = C[-\tau, 0]$ (infinite dimensional state space)

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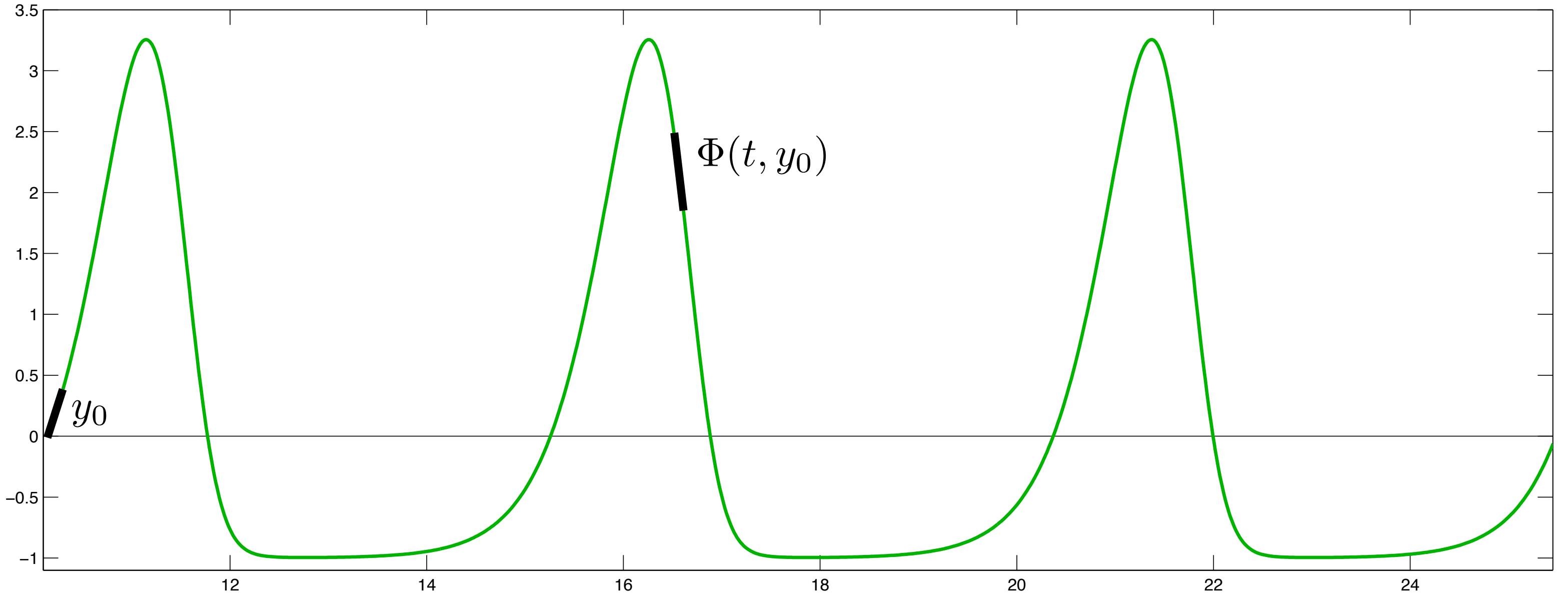


ex: $y'(t) = -\frac{12}{5} y(t-1)[1 + y(t)]$

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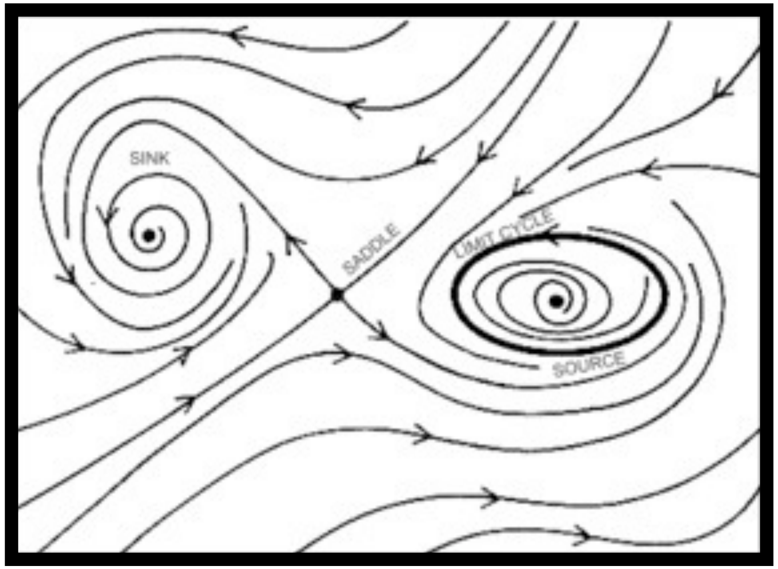


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- Time periodic solutions.
- Connecting orbits.
- Global attractors.

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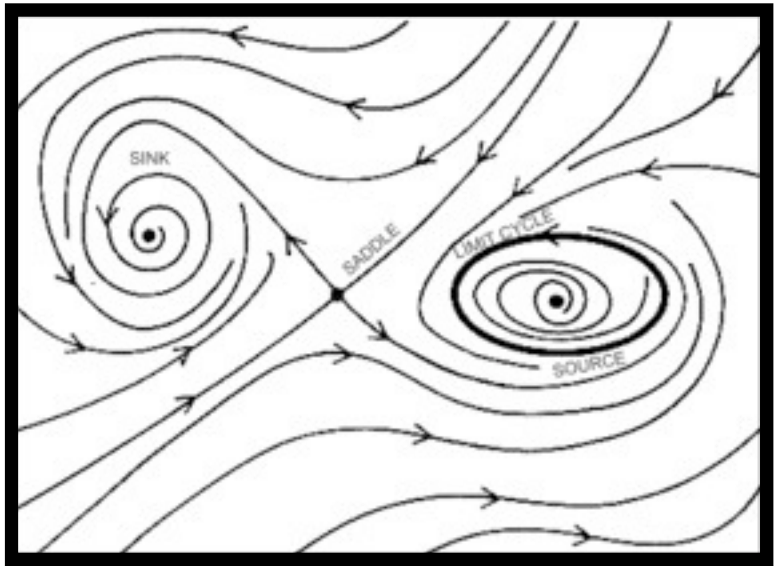


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A standard approach is to get insight from numerical simulations to formulate new conjectures, and then attempt to prove the conjectures using pure mathematical techniques only. Actually, this strong dichotomy need not exist in the context of dynamical systems, as the strength of numerical analysis and functional analysis can be combined to prove, in a **rigorous mathematical sense**, the existence of equilibria, periodic solutions, connecting orbits.... and even chaotic dynamics !

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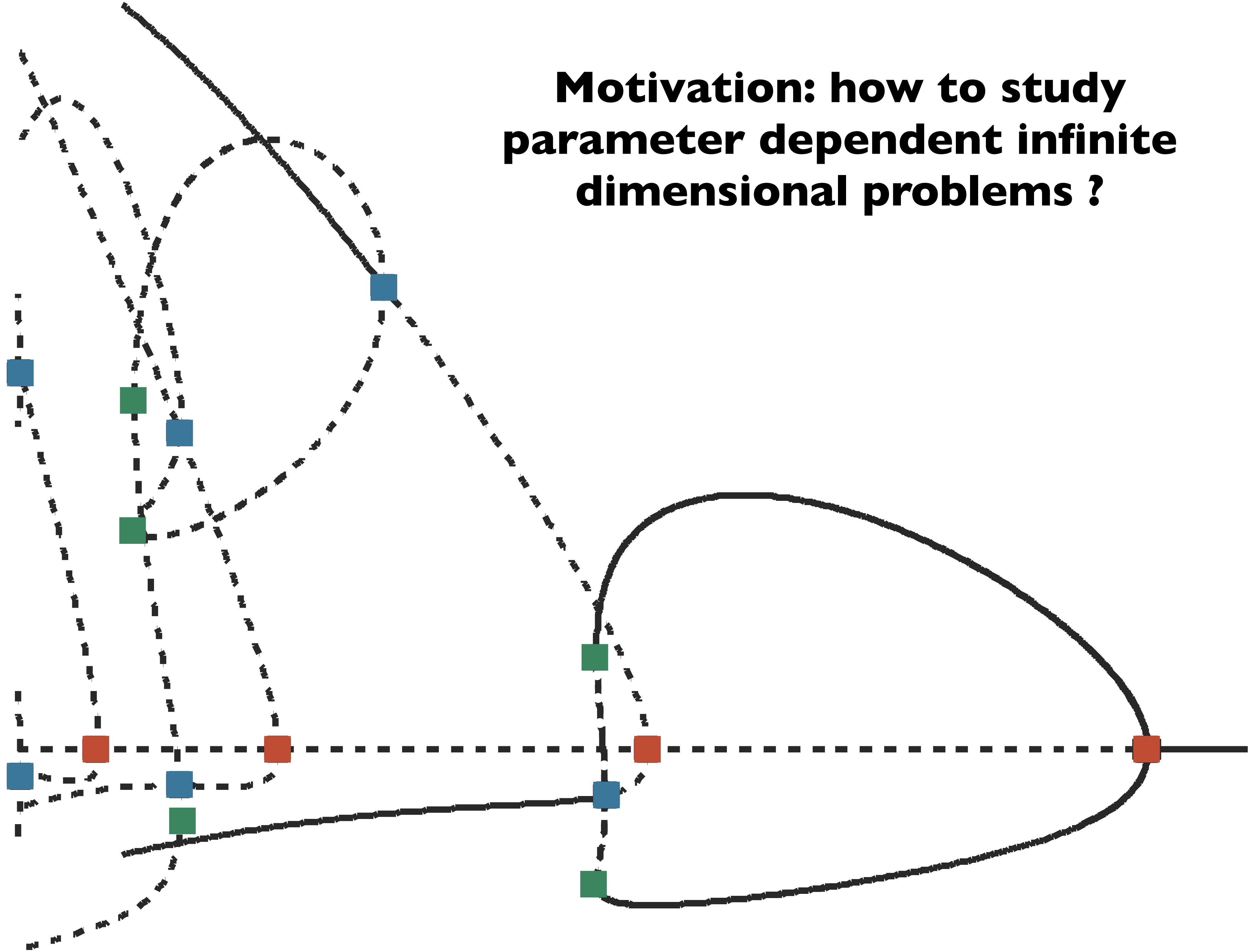
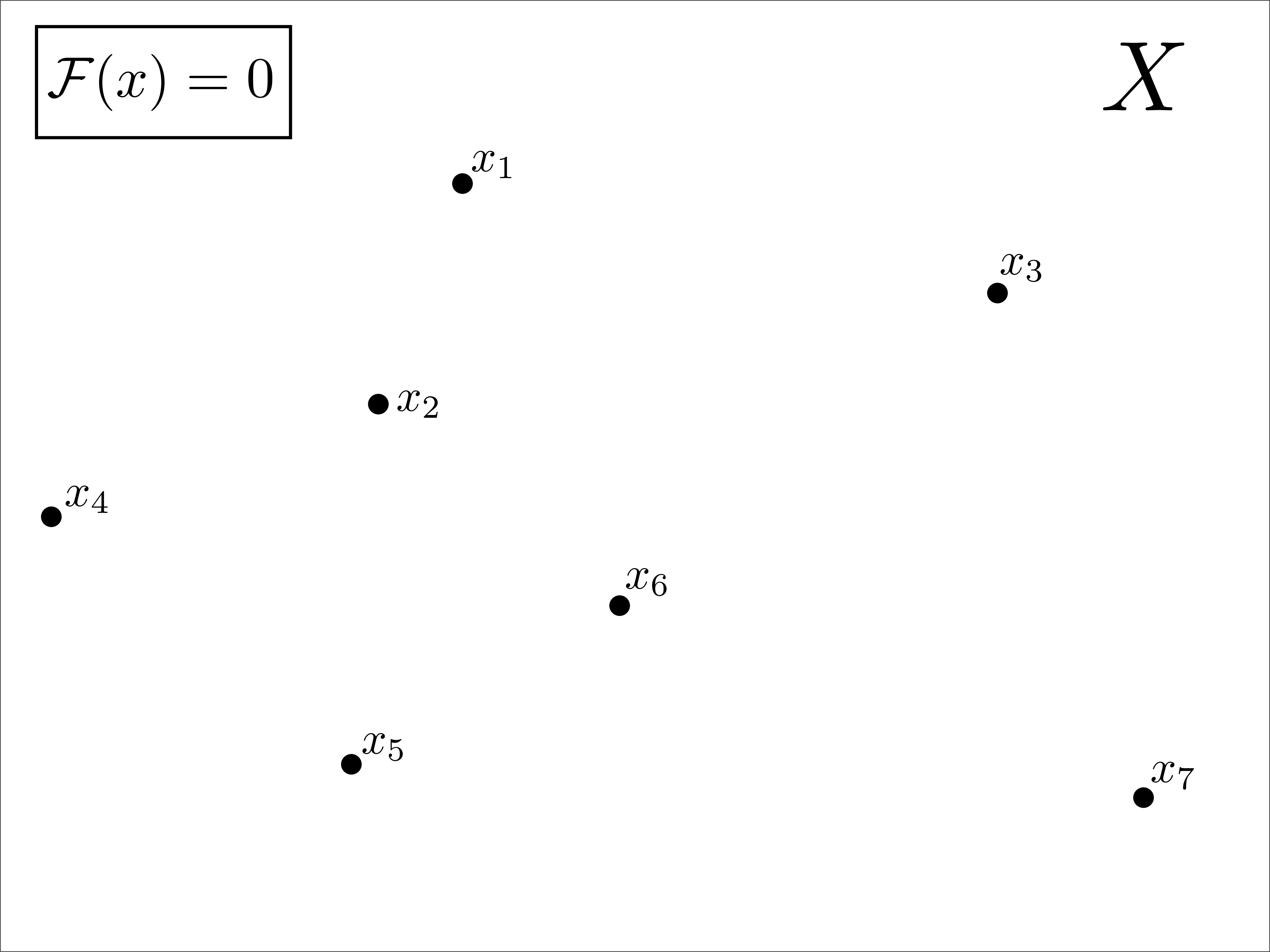
## Rigorous computations

**The goal of rigorous computations is to construct algorithms that provide an approximate solution to the problem together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.**

# Motivation: how to study parameter dependent infinite dimensional problems ?





$\mathcal{F}(x) = 0$
$X$
$x_1$
$x_3$
$x_2$
$x_4$
$x_6$
$x_5$
$x_7$

$$\mathcal{F}(x) = 0$$

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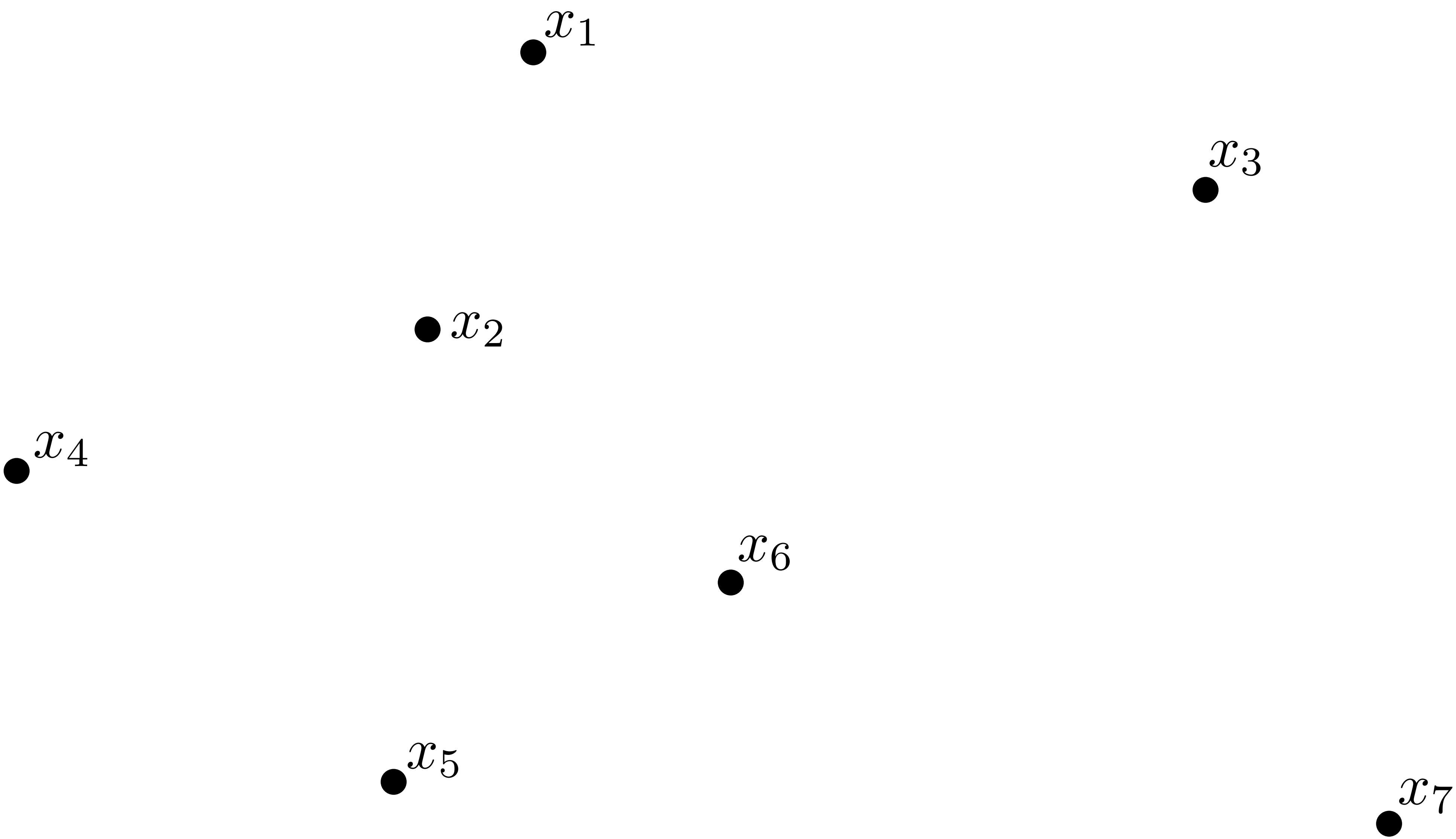
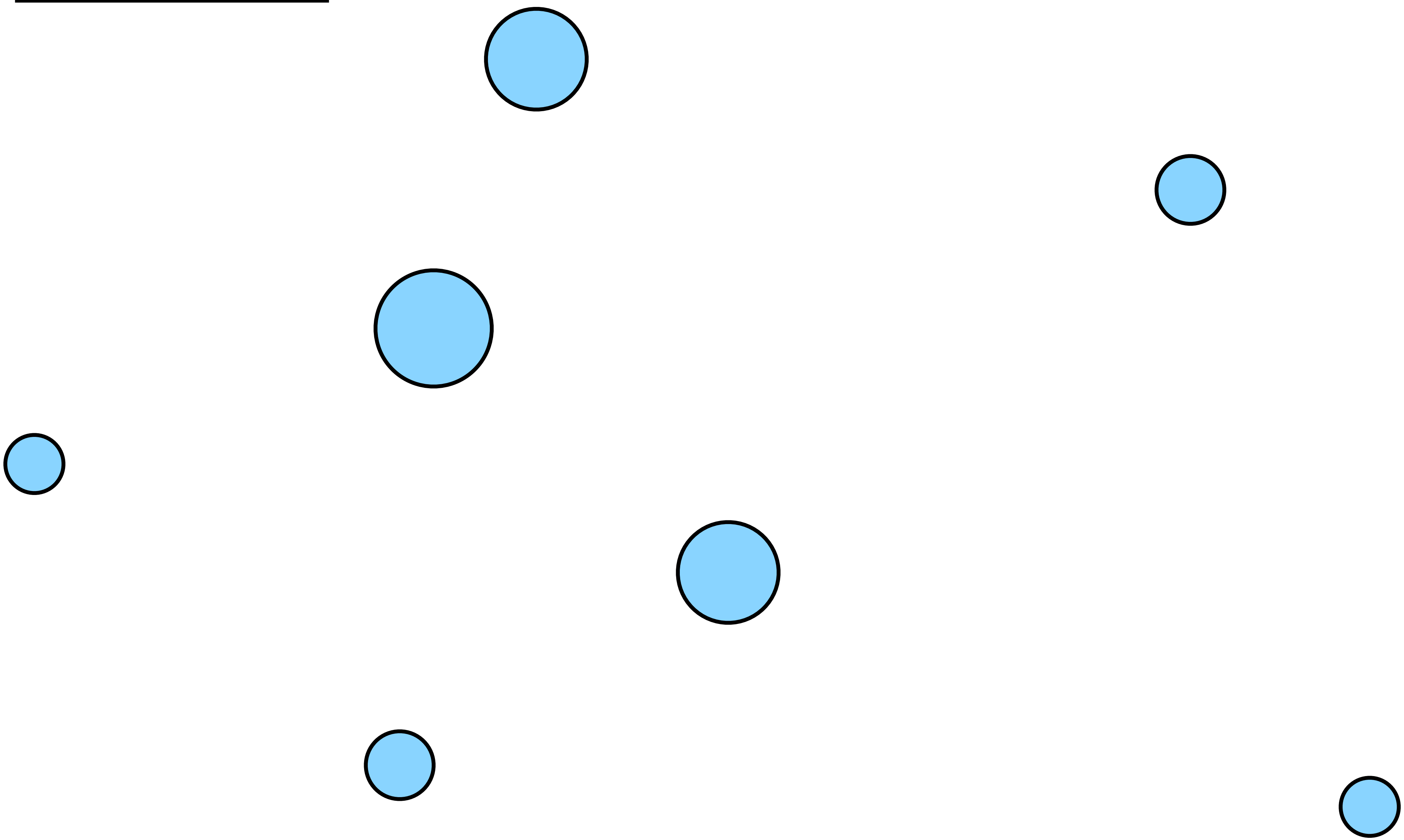



**Often impossible to compute exactly !**

$$\boxed{\mathcal{F}(x) = 0}$$

$X$



**Alternative: find small balls in which it is demonstrated (in a mathematically rigorous sense) that a unique solution exists.**

# Rigorous Computations

## (Ingredients)

1. Smoothness of the solutions
2. Banach space of algebraically decaying sequences
3. Finite dimensional Galerkin projection
4. Bounds on the truncation error terms (Analytic estimates)
5. Fixed point theory, Uniform contraction principle
6. Numerical analysis (continuation, Fast Fourier transform)
7. Interval Arithmetic

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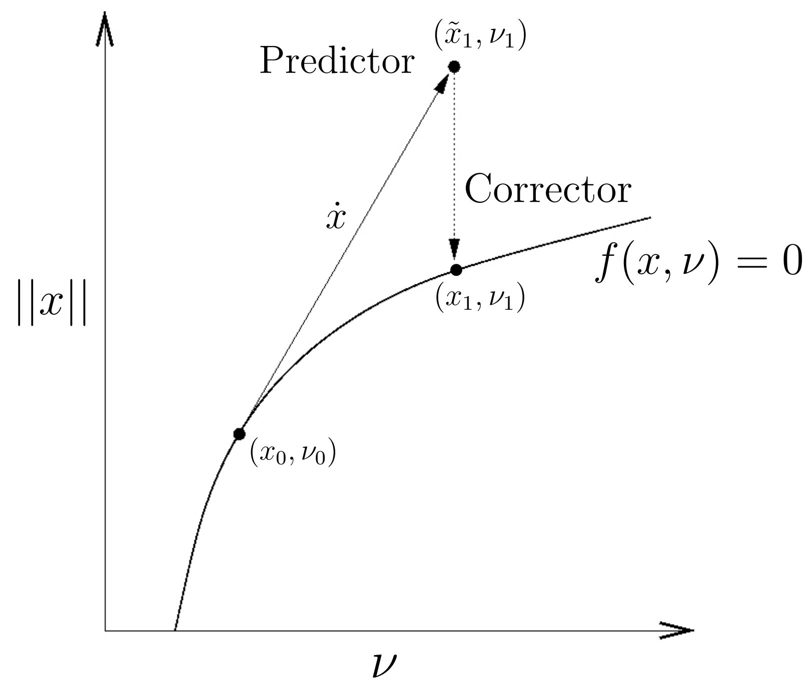
**Continuation**

**(Predictor-Corrector Algorithm)**



# Rigorous Computations

$$F(u,\nu) = 0$$

(Differential Equation)

spectral method $\rightsquigarrow$

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$x$ : modes

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Knowledge about regularity $\rightsquigarrow$

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Consider $\bar{x}$ such that $f^{(m)}(\bar{x},\nu_0)\approx 0$. **Galerkin approximation**

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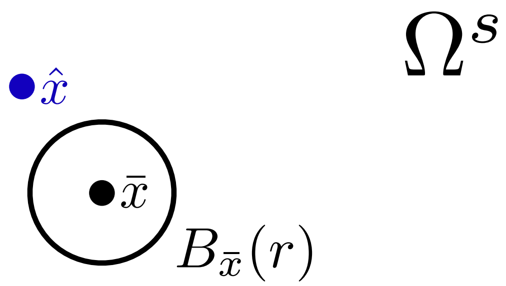
$$T_\nu : \Omega^s \to \Omega^s$$
$$T_\nu(x) = x - Jf(x,\nu)$$
$$J \approx D_x f(\bar{x},\nu_0)^{-1}$$

The chances of contracting a small set B around $\bar{x}$ depends on the magnitude of the eigenvalues of $J$.

**Q:** How to find a ball $B_{\bar{x}}(r)$ such that $T_\nu \colon B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction?



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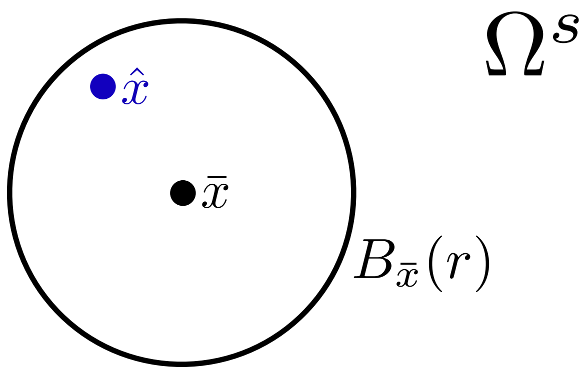
$$B_{\bar{x}}(r) = \bar{x} + B(r)$$



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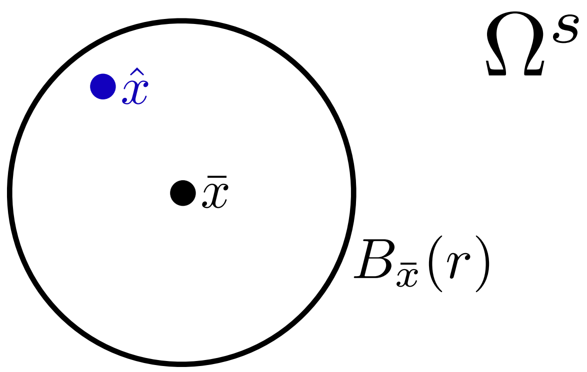
Ball of radius r centered at 0 in the space $\Omega^s$



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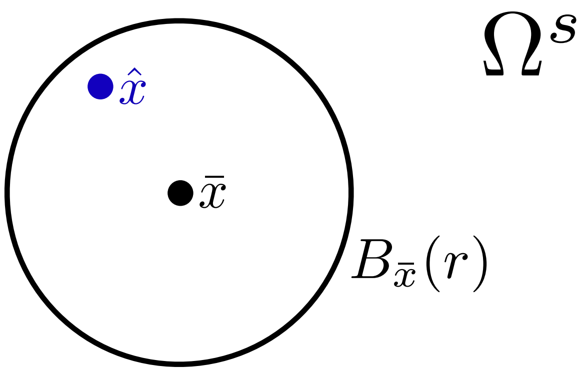


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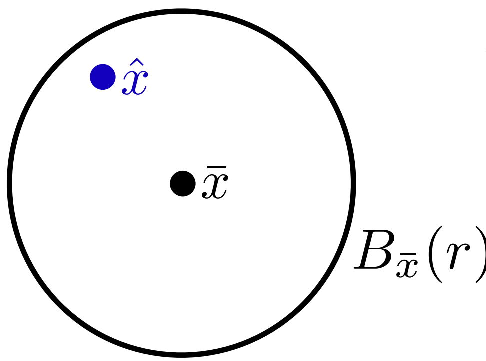
$B(r)$ → Ball of radius r centered at 0 in the space $\Omega^s$

**A:** **Radii polynomials** $\{p_k(r)\}_k$ : upper bounds satisfying

$$\left| \left[ T_\nu(\bar{x}) - \bar{x} \right]_k \right| + \sup_{b,c \in B(r)} \left| \left[ D_x T_\nu(\bar{x} + b) c \right]_k \right| - \frac{r}{\omega_k^s} \le p_k(r)$$

$\Omega^s$

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**Lemma:** If there exists $r > 0$ such that $p_k(r) < 0$ for all $k$, then there is a unique $\hat{x} \in B_{\bar{x}}(r)$ s.t. $f(\hat{x}, \nu) = 0$.

*proof.* Banach fixed point theorem.

# Analytic estimates to construct the polynomials

Suppose there exist $A_1, A_2, \dots, A_n$ such that for every $j \in \{1, \dots, n\}$ and every $\boldsymbol{k} \in \mathbb{Z}^d$, we have that

$$\left| c_{\boldsymbol{k}}^{(j)} \right| \le \frac{A_j}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}},$$

$$\omega_{\boldsymbol{k}}^{\boldsymbol{s}} = |k_1|^{s_1} \cdots |k_d|^{s_d}$$

Then, for any $\boldsymbol{k} \in \mathbb{Z}^d$, we get that

$$\left| \left( c^{(1)} * \cdots * c^{(n)} \right)_{\boldsymbol{k}} \right| \le \left( \prod_{j=1}^{n} A_j \right) \frac{\alpha_{\boldsymbol{k}}^{(n)}}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}.$$

*Proof.*

$$\begin{aligned}
\left| \left( c^{(1)} * \cdots * c^{(n)} \right)_{\boldsymbol{k}} \right| &= \left| \sum_{\substack{\boldsymbol{k}^1 + \cdots + \boldsymbol{k}^n = \boldsymbol{k} \\ \boldsymbol{k}^1, \dots, \boldsymbol{k}^n \in \mathbb{Z}^d}} c_{\boldsymbol{k}^1}^{(1)} \cdots c_{\boldsymbol{k}^n}^{(n)} \right| \le \sum_{\substack{\boldsymbol{k}^1 + \cdots + \boldsymbol{k}^n = \boldsymbol{k} \\ \boldsymbol{k}^1, \dots, \boldsymbol{k}^n \in \mathbb{Z}^d}} \frac{A_1}{\omega_{\boldsymbol{k}^1}^{\boldsymbol{s}}} \cdots \frac{A_n}{\omega_{\boldsymbol{k}^n}^{\boldsymbol{s}}} \\
&= \left( \prod_{j=1}^{n} A_j \right) \left( \sum_{\substack{\boldsymbol{k}^1 + \cdots + \boldsymbol{k}^n = \boldsymbol{k} \\ \boldsymbol{k}^1, \dots, \boldsymbol{k}^n \in \mathbb{Z}^d}} \frac{1}{\omega_{\boldsymbol{k}^1}^{\boldsymbol{s}} \cdots \omega_{\boldsymbol{k}^n}^{\boldsymbol{s}}} \right) \\
&= \left( \prod_{j=1}^{n} A_j \right) \left( \sum_{\substack{\boldsymbol{k}^1 + \cdots + \boldsymbol{k}^n = \boldsymbol{k} \\ \boldsymbol{k}^1, \dots, \boldsymbol{k}^n \in \mathbb{Z}^d}} \prod_{j=1}^{d} \frac{1}{\omega_{k_j^1}^{s_j} \cdots \omega_{k_j^n}^{s_j}} \right) \\
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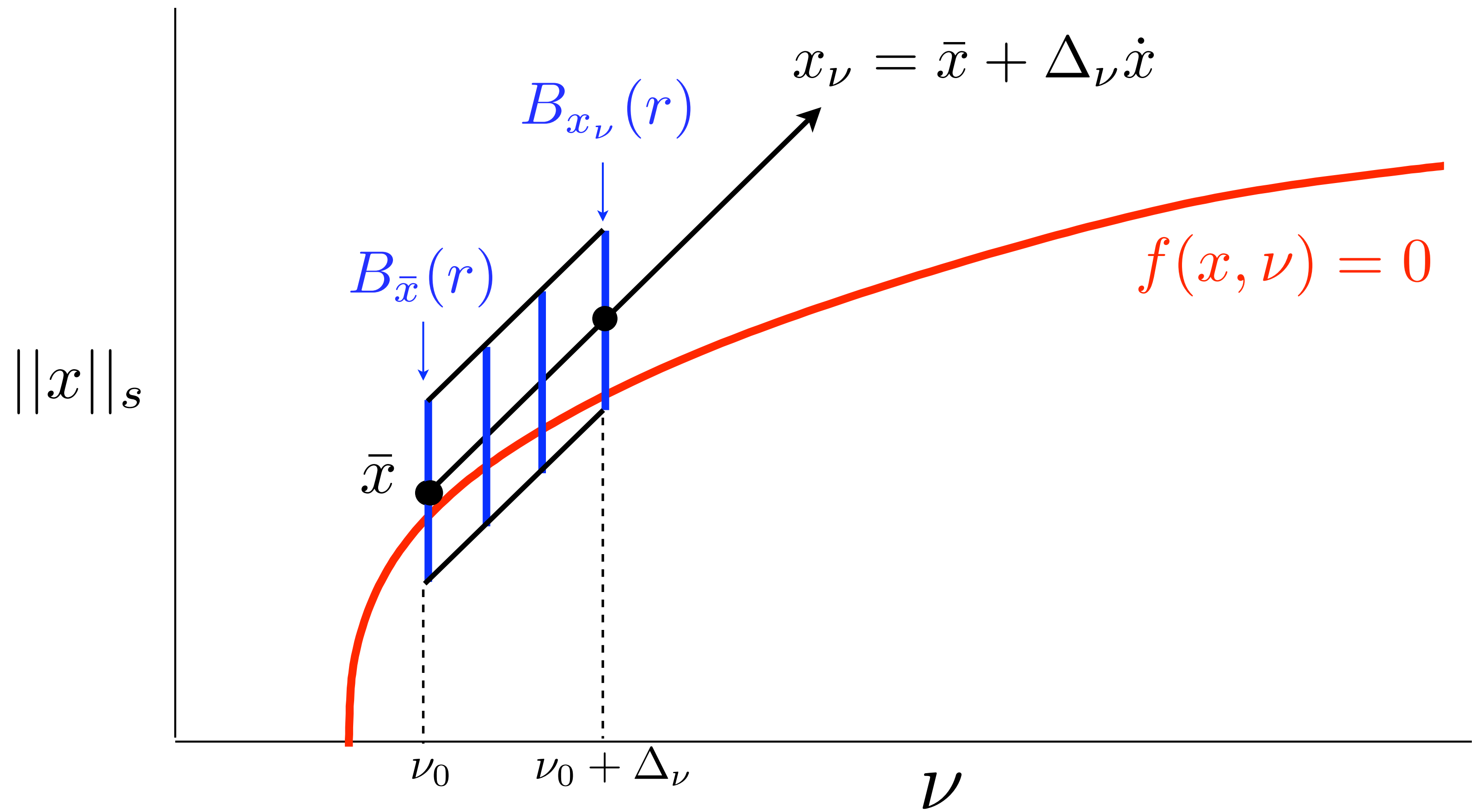
&\le \left( \prod_{j=1}^{n} A_j \right) \prod_{j=1}^{d} \frac{\alpha_{k_j}^{(n)}}{\omega_{k_j}^{s_j}} = \left( \prod_{j=1}^{n} A_j \right) \frac{\alpha_{\boldsymbol{k}}^{(n)}}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}.
\end{aligned}$$

**M. Gameiro & J.-P. L.** **Analytic estimates and rigorous continuation for equilibria of higher-dimensional PDEs.** ***Journal of Differential Equations*****, 2010.**

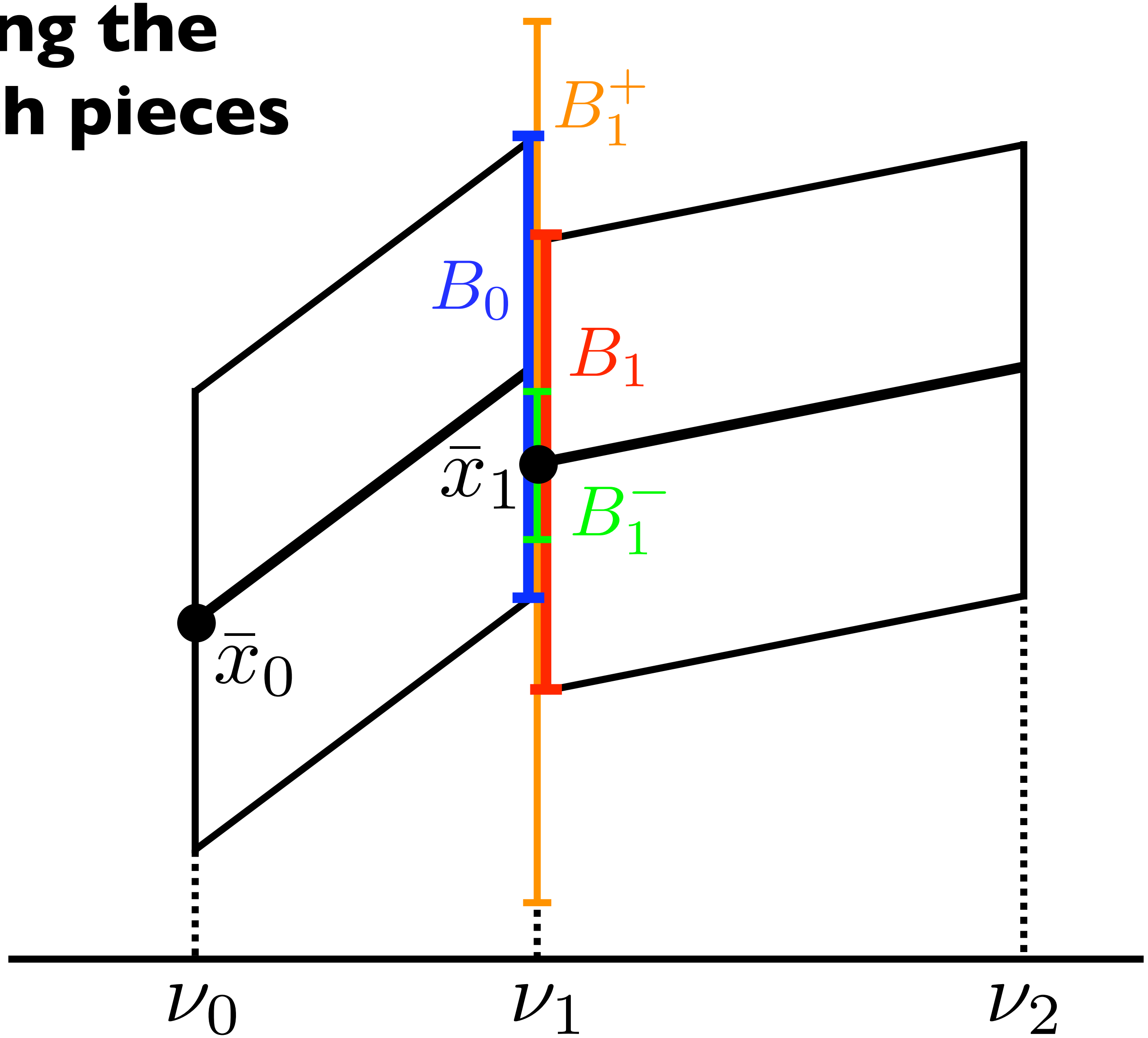
**Radii polynomials** $\{p_{\boldsymbol{k}}(r,\Delta_\nu)\}$ $\rightsquigarrow$ Verifying the uniform contraction principle.

$\exists\ r>0$ **s.t.** $p_{\boldsymbol{k}}(r,\Delta_\nu)<0, \forall \boldsymbol{k} \Longrightarrow T$ : uniform contraction on $[\nu_0,\nu_0+\Delta_\nu]$

# The rigorous computational method



## Gluing the smooth pieces



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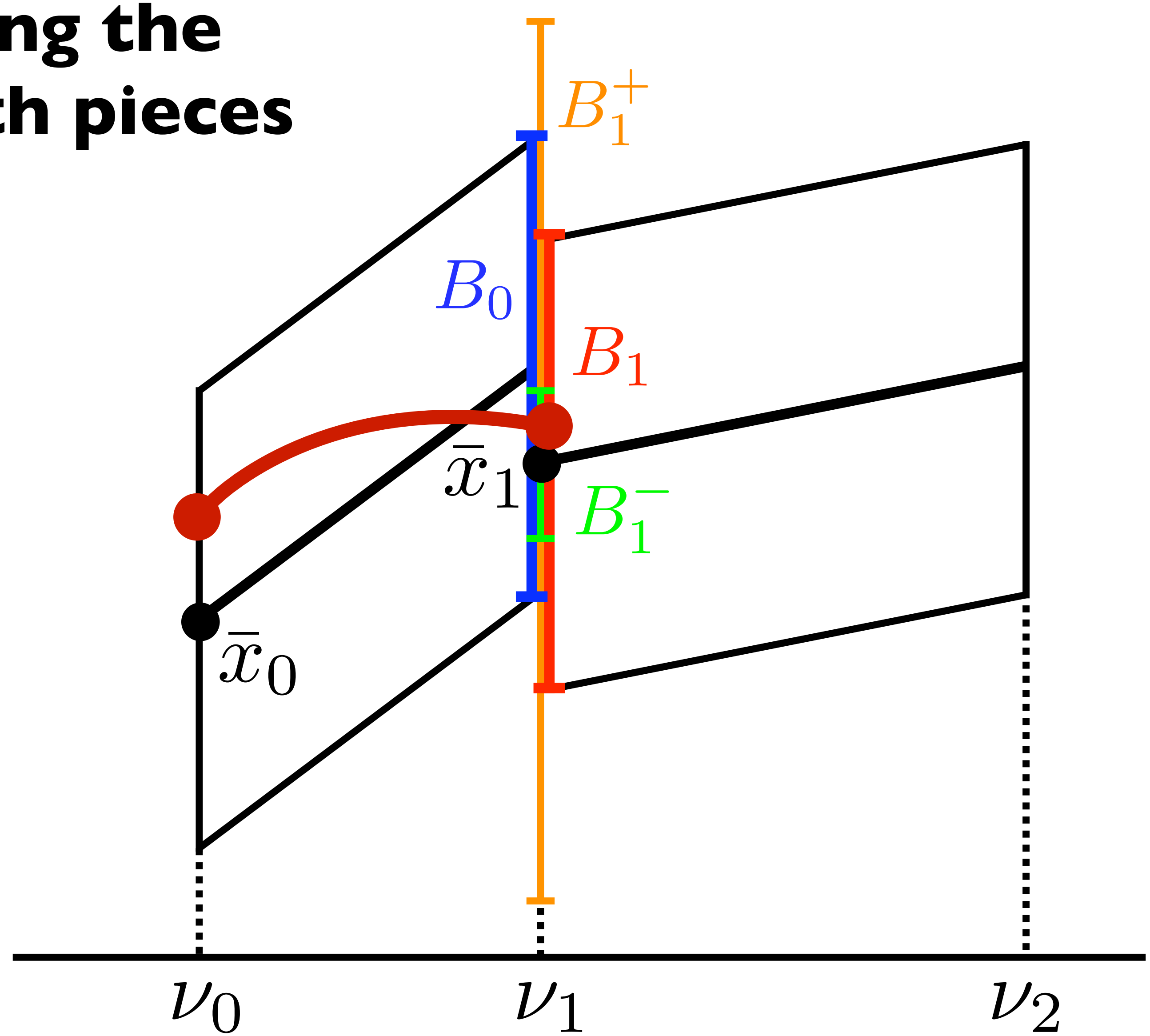


$$\{(x,\nu) \mid f(x,\nu) = 0,\ \nu \in [\nu_0, \nu_1]\}$$

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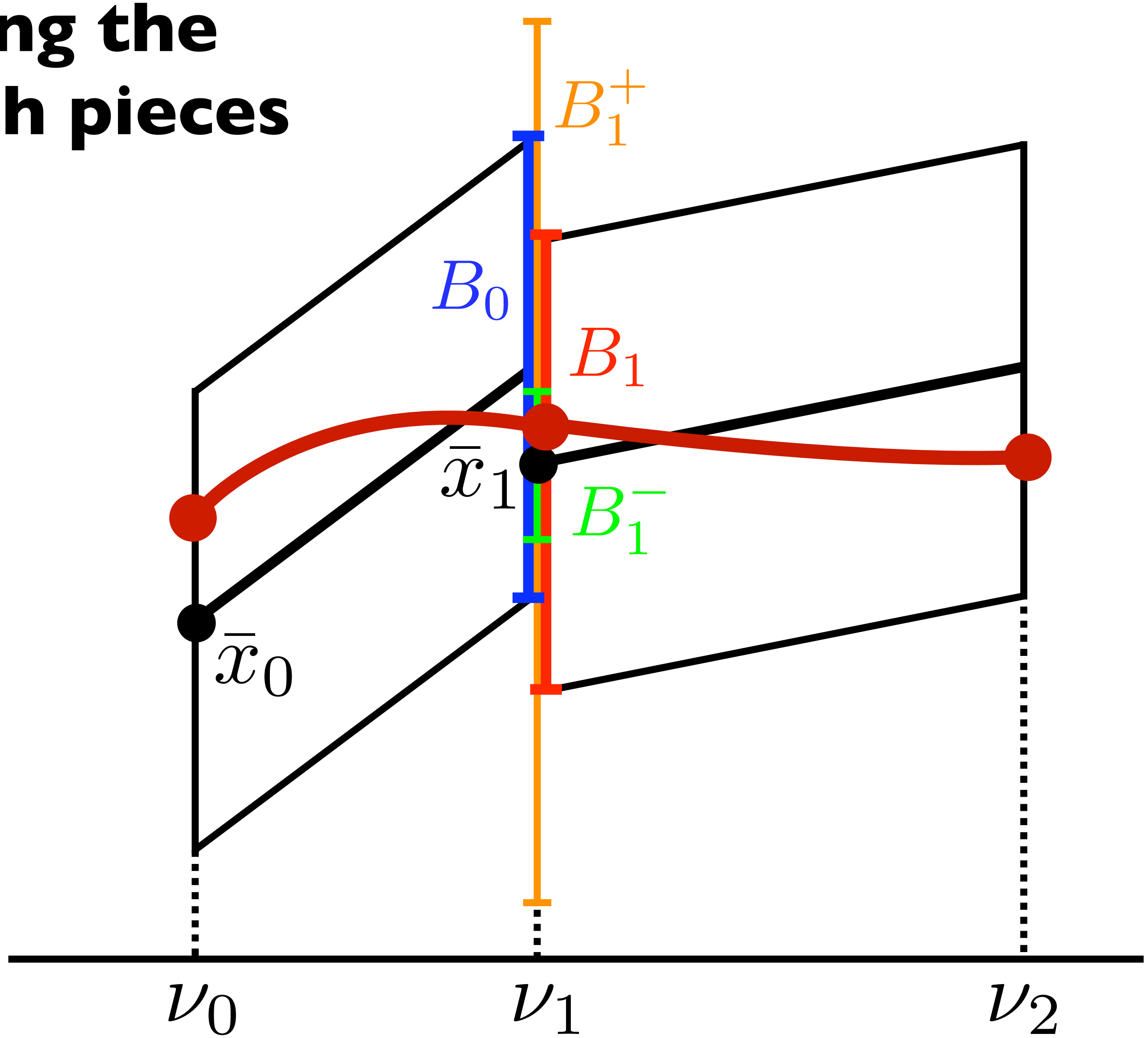


$$\{(x,\nu) \mid f(x,\nu)=0,\ \nu \in [\nu_0,\nu_2]\}$$

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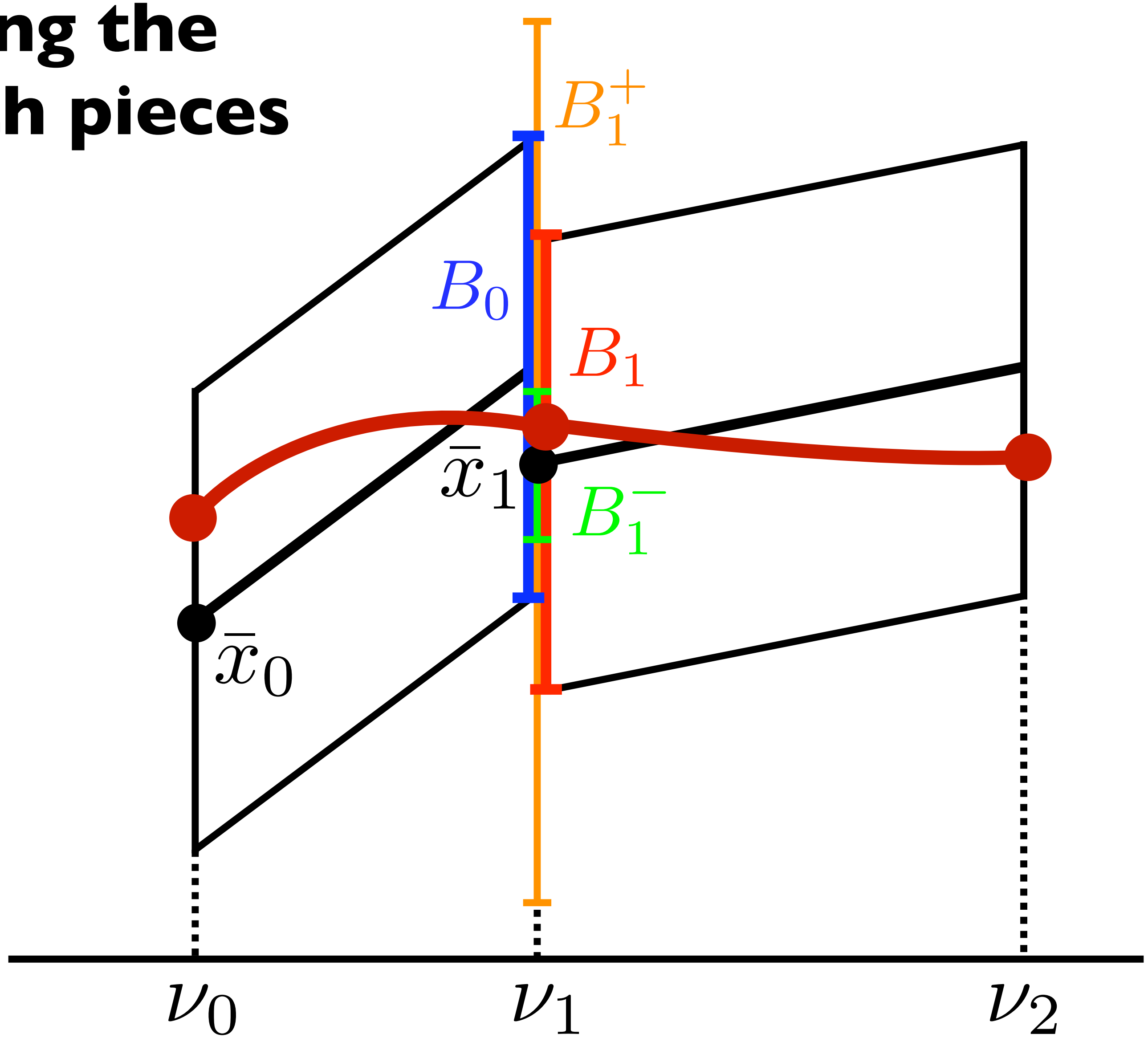


$$\{(x,\nu) \mid f(x,\nu) = 0,\ \nu \in [\nu_0, \nu_2]\}$$

- **Global smooth curves of solutions.**
- **Local uniqueness by the Banach fixed point theorem.**
- **Proof of non existence of secondary bifurcations along the curves.**

# Applications

- **Initial value problems of ODEs** (Chebyshev in time)
- **Boundary value problems of ODEs** (Chebyshev in time)
- **Periodic solutions of ODEs** (Fourier in time)
- **Connecting orbits of ODEs** (Chebyshev in time + parameterization of invariant manifolds using power series)
- **Equilibria of PDEs** (Fourier in space)
- **Periodic solutions of delay differential equations** (Fourier in time)
- **Minimizers of action functionals** (Chebyshev in time)
- **Periodic solutions of PDEs** (Fourier in space and in time)

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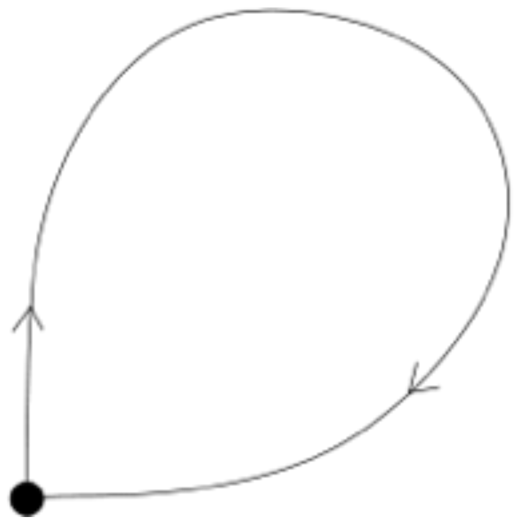
# 1. Homoclinic and heteroclinic orbits of ODEs (traveling waves)

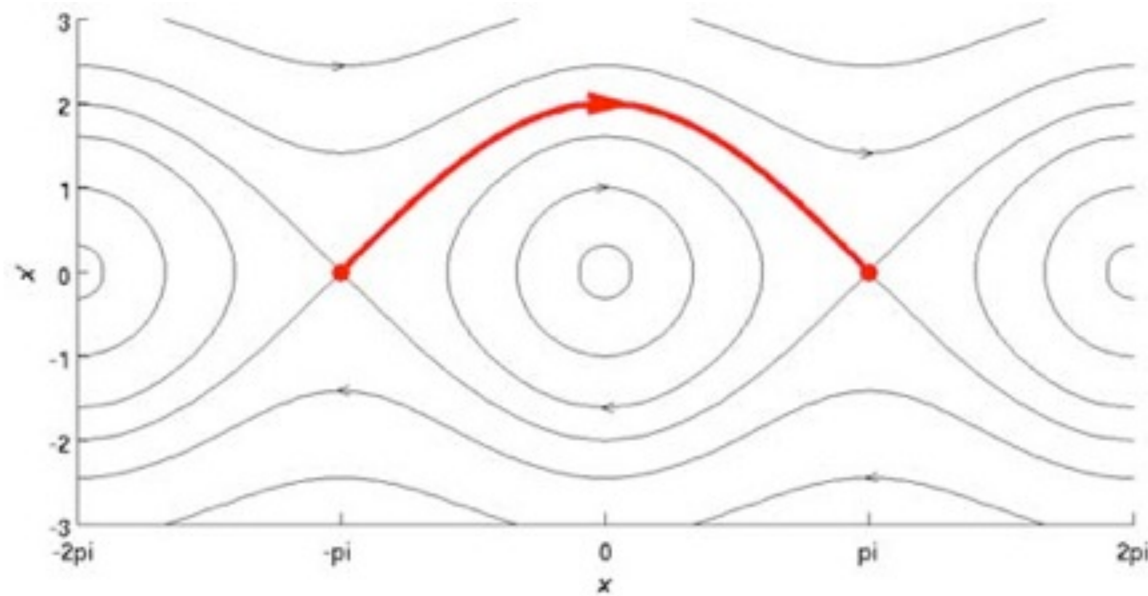
**ODEs** $$\frac{dx}{dt} = f(x)$$

$$\lim_{t \to \pm\infty} x(t) = x^{\pm} \in \mathbb{R}^n$$



$$x^{+} = x^{-}$$

**homoclinic orbit**

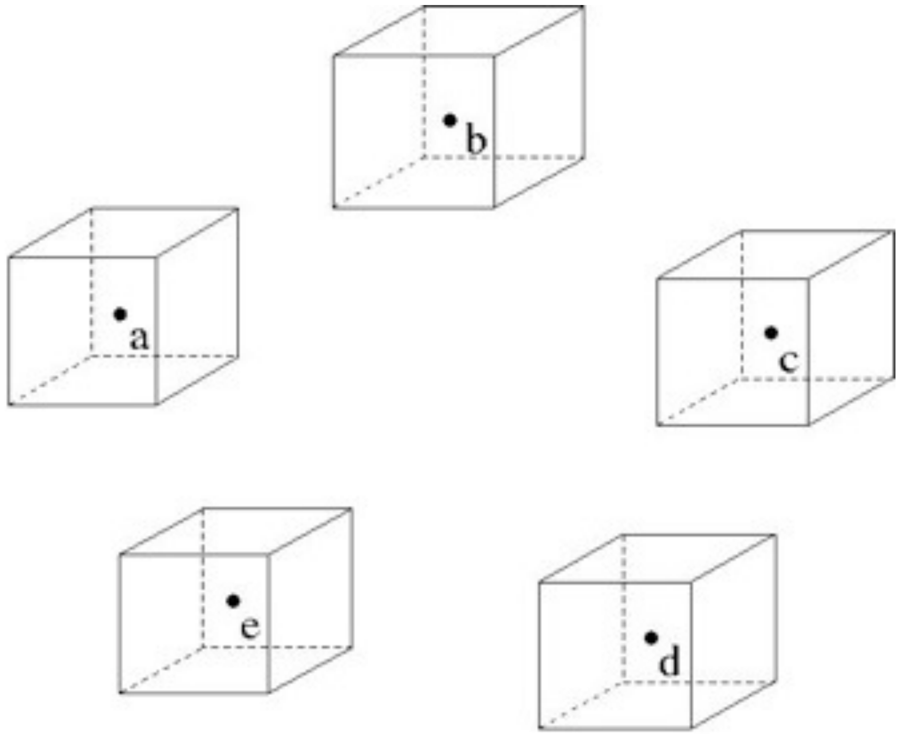



$$x^{+} \neq x^{-}$$

**heteroclinic orbit**

# Rigorous Computations

## Connecting Orbits




**Compute a set of equilibria.**

# Rigorous Computations

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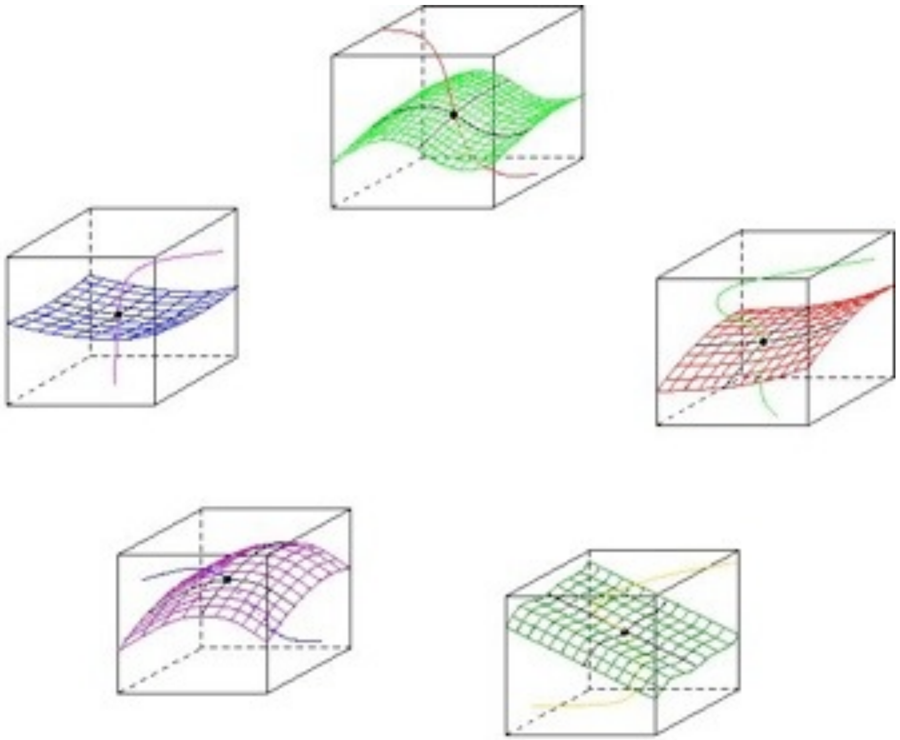


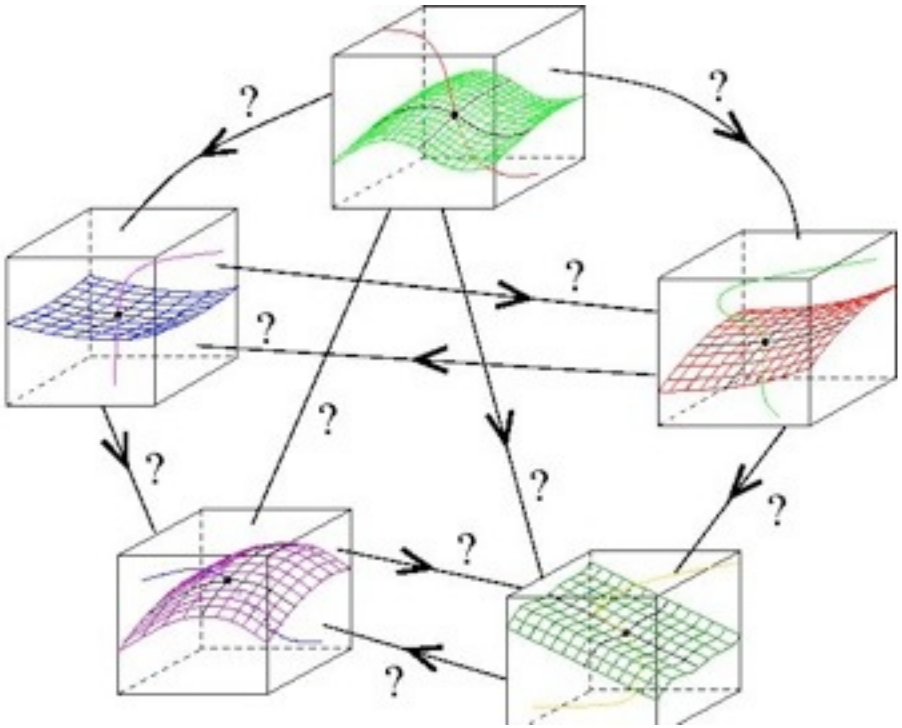
**Compute a set of equilibria.**

**Local representation of the invariant manifolds.**

**Parameterization method**

# Rigorous Computations

## Connecting Orbits



**Compute a set of equilibria.**

**Local representation of the invariant manifolds.**

Parameterization method

**Connecting orbits between the equilibria?**

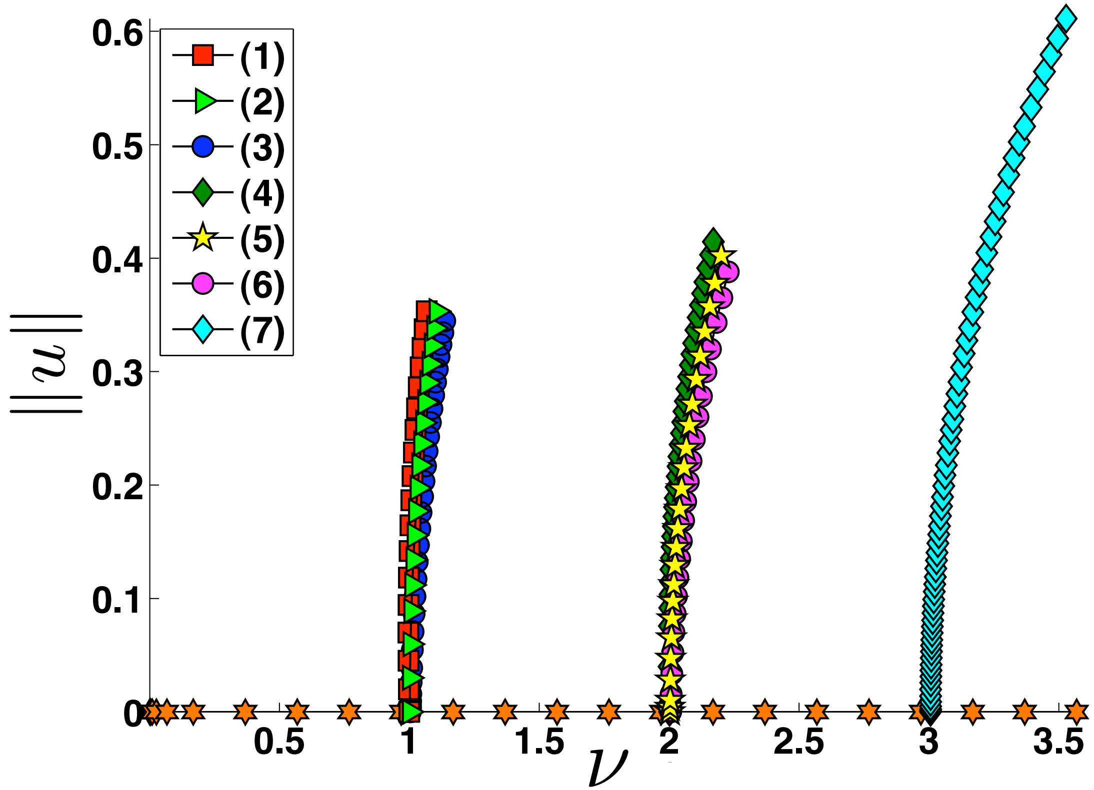
Boundary value problem
Chebyshev series
Radii polynomials

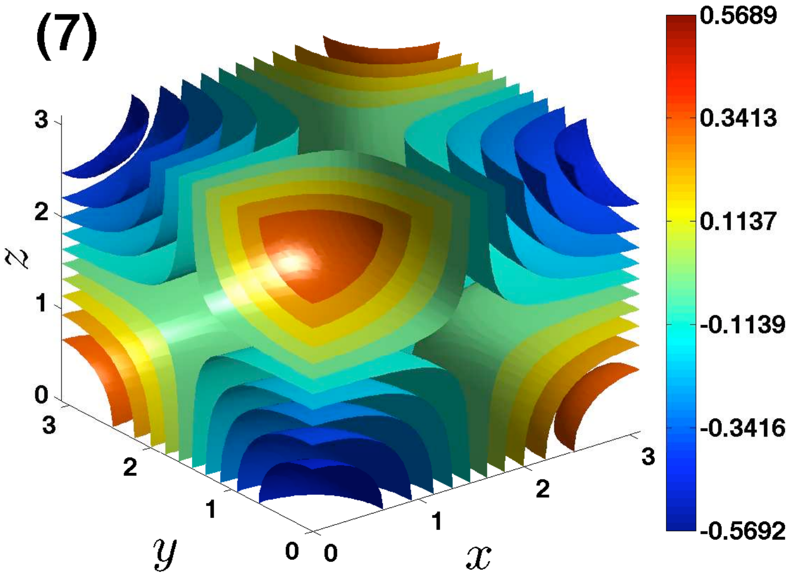
# 2. Equilibria of PDEs

**Cahn-Hilliard 3D**

$$\begin{cases} u_t = -\Delta(\frac{1}{\nu}\Delta u + u - u^3), & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \Delta u}{\partial n} = 0, & \text{on } \partial\Omega \end{cases}$$

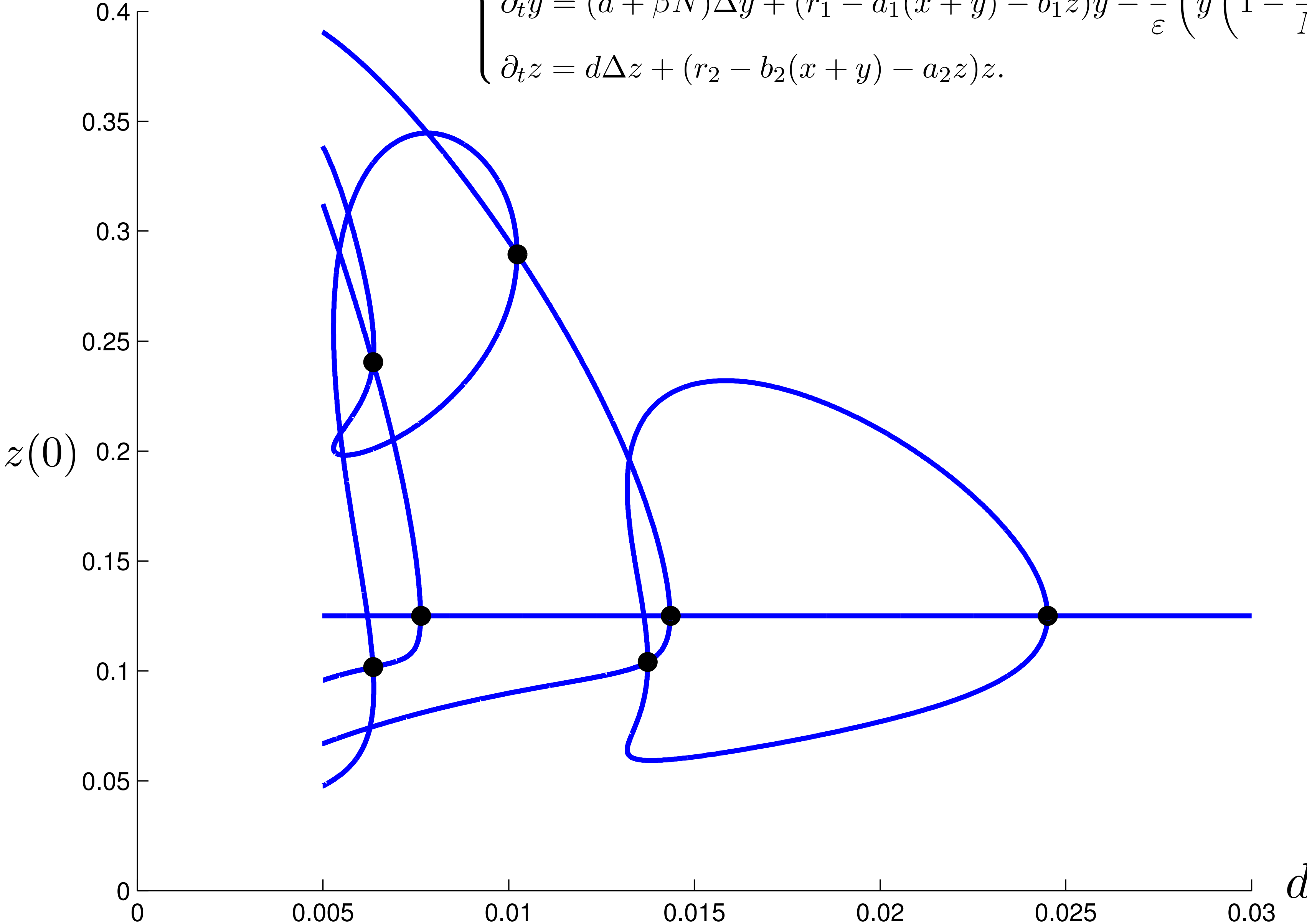
$$\Omega = [0,\pi] \times [0, \frac{\pi}{1.001}] \times [0, \frac{\pi}{1.002}]$$



# Systems of reaction-diffusion PDEs

$$\begin{cases} \partial_t x = d\Delta x + (r_1 - a_1(x+y) - b_1 z)x + \dfrac{1}{\varepsilon}\left(y\left(1 - \dfrac{z}{N}\right) - x\dfrac{z}{N}\right), \\ \partial_t y = (d + \beta N)\Delta y + (r_1 - a_1(x+y) - b_1 z)y - \dfrac{1}{\varepsilon}\left(y\left(1 - \dfrac{z}{N}\right) - x\dfrac{z}{N}\right), \\ \partial_t z = d\Delta z + (r_2 - b_2(x+y) - a_2 z)z. \end{cases}$$



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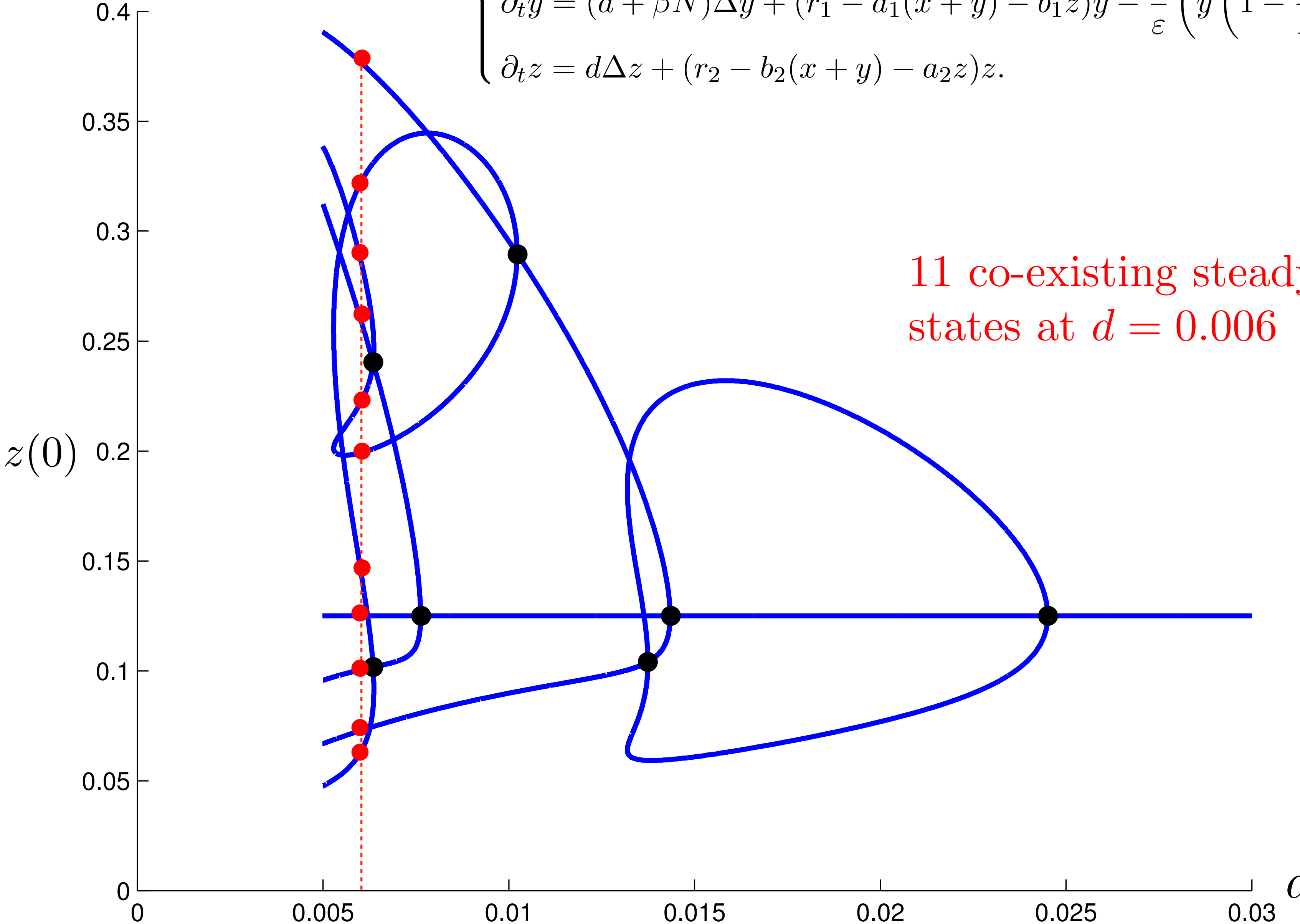
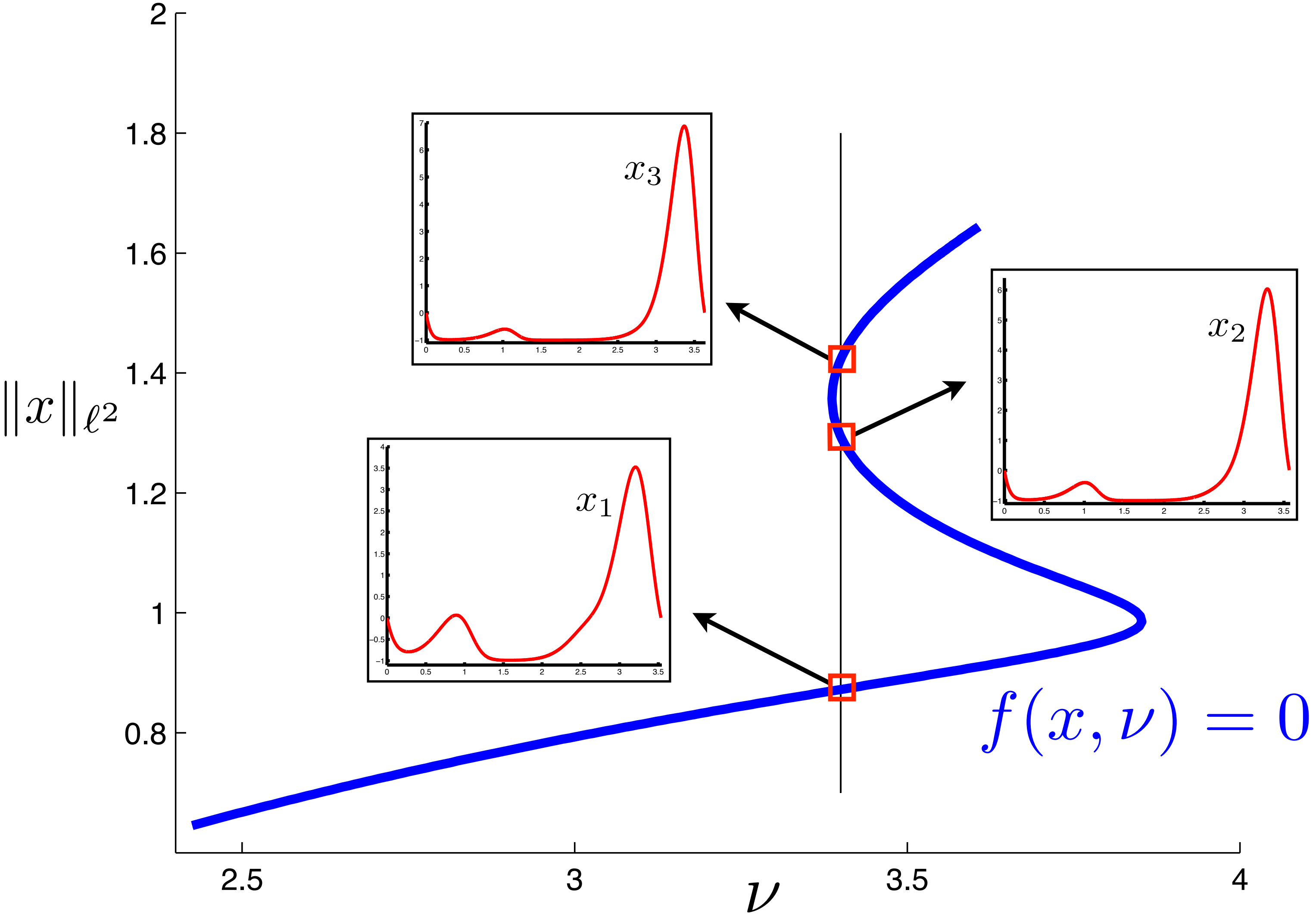
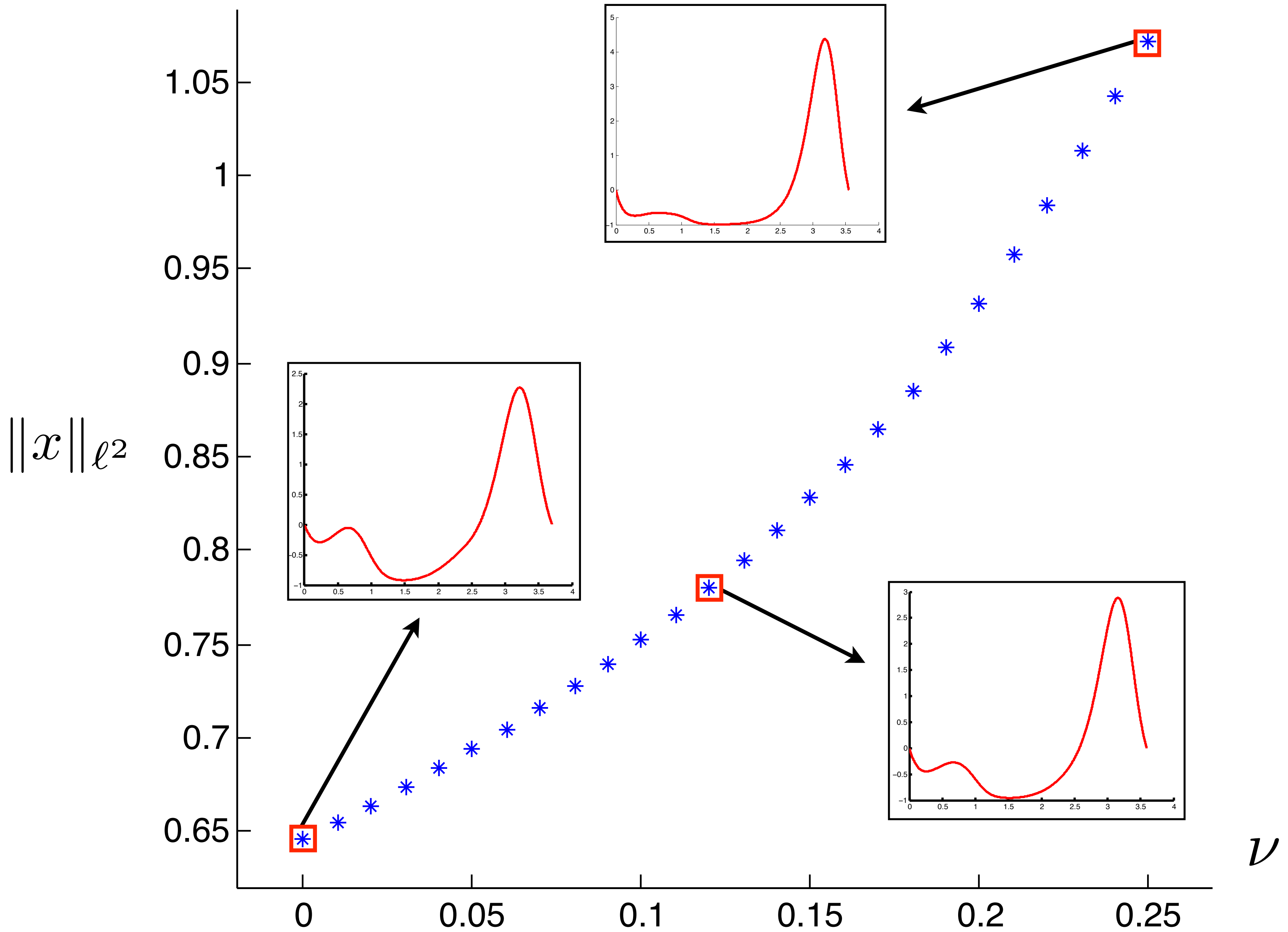


# 3. Periodic solutions of delay equations

$$y'(t) = \mathcal{F}\left(y(t), y(t-\tau_1), \ldots, y(t-\tau_d)\right),$$




$$y'(t) = -\nu\left[y(t-\tau_1) + y(t-\tau_2)\right]\left[1 + y(t)\right],$$




$$y'(t) = -\left[2.425y(t-\tau_1) + 2.425y(t-\tau_2) + \nu y(t-\tau_3)\right]\left[1 + y(t)\right],$$

# 4. Minimizers of action functionals

## Ginzburg–Landau energy: a model of superconductivity

$$G = G(\phi, a) = \frac{1}{2d}\int_{-d}^{d}\left(\phi^2(\phi^2 - 2) + \frac{2(\phi')^2}{\kappa^2} + 2\phi^2 a^2 + 2(a' - h_e)^2\right)dt.$$

$\phi > 0$ : measures the density of superconducting electrons

$a$ : magnetic field potential

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**<u>Parameters</u>**

$d$ : size of the superconducting material

$h_e$ : external magnetic field

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$d$ : size of the superconducting material

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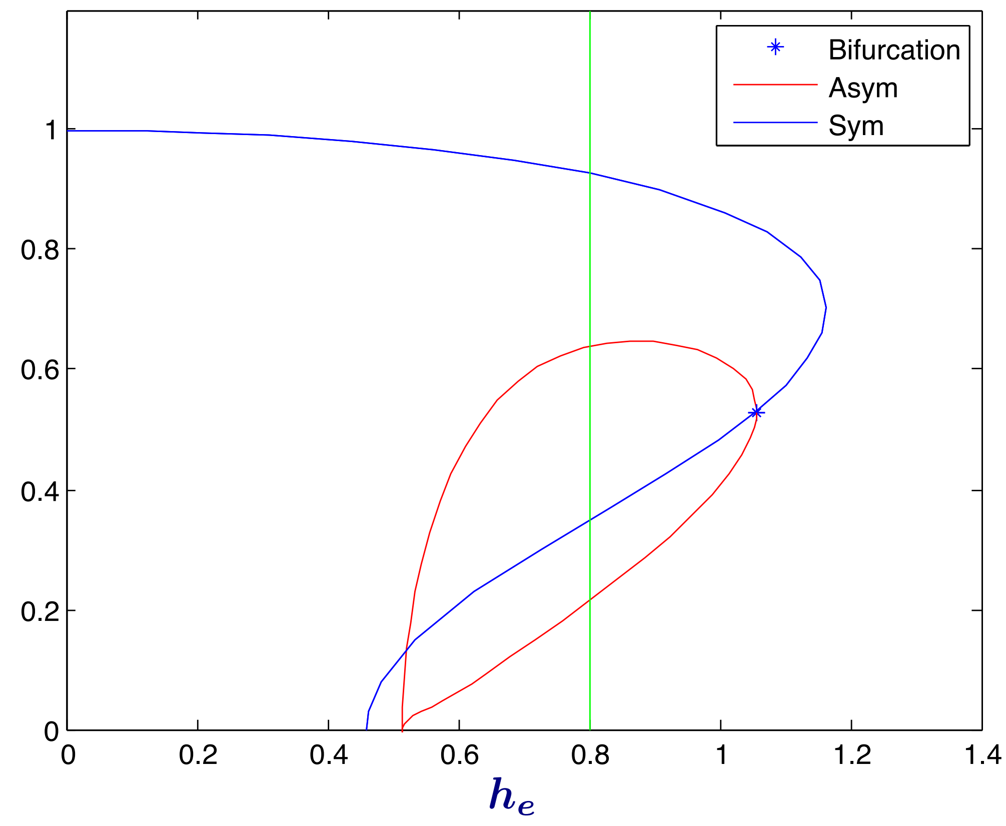
$\kappa = 0.3,\ d = 4$




**Co-existence of nontrivial solutions**

# 5. Periodic orbits of PDEs

## Kuramoto-Sivashinski equation

$$
\text{(KS)} \quad \begin{cases} u_t = -\nu u_{yyyy} - u_{yy} + 2uu_y \\ u(t,y) = u(t,y+2\pi), \quad u(t,-y) = -u(t,y) \end{cases}
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**Goal: propose an method (based on spectral methods and fixed point theory) to rigorously compute time periodic solutions of PDEs.**

Letting $L = \frac{2\pi}{p}$, the time-periodic solutions of period $p$ of (KS) can be expanded using the Fourier expansion

$$u(t,y) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} c_{\boldsymbol{k}}\psi_{\boldsymbol{k}}, \qquad \text{where for } \boldsymbol{k} = (k_1,k_2) \in \mathbb{Z}^2, \quad \psi_{\boldsymbol{k}} = e^{iLk_1t}e^{ik_2y}.$$

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$$x_{\boldsymbol{k}} = \begin{cases} L, & \boldsymbol{k} = (0,0) \\ b_{\boldsymbol{k}}, & \boldsymbol{k} = (0,k_2), \quad k_2 \neq 0 \\ \begin{pmatrix} a_{\boldsymbol{k}} \\ b_{\boldsymbol{k}} \end{pmatrix}, & \boldsymbol{k} = (k_1,k_2), \quad k_1 \neq 0 \text{ and } k_2 \neq 0. \end{cases}$$

$$a_{\boldsymbol{k}} \stackrel{\text{def}}{=} Re(c_{\boldsymbol{k}}) \quad \text{and} \quad b_{\boldsymbol{k}} \stackrel{\text{def}}{=} Im(c_{\boldsymbol{k}}).$$

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Plugging the space-time Fourier expansion into (KS) results in solving, for all $\boldsymbol{k} \in \mathbb{Z}^2$

$$h_{\boldsymbol{k}} \overset{\text{def}}{=} \mu_{\boldsymbol{k}}c_{\boldsymbol{k}} - 2\sum_{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}} i\boldsymbol{k}_2^1 c_{\boldsymbol{k}^1}c_{\boldsymbol{k}^2} = \mu_{\boldsymbol{k}}c_{\boldsymbol{k}} - k_2 i \sum_{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}} c_{\boldsymbol{k}^1}c_{\boldsymbol{k}^2} = 0,$$

where $\mu_{\boldsymbol{k}} = \mu_{k_1,k_2} \overset{\text{def}}{=} ik_1L + \nu k_2^4 - k_2^2$.

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**Functions**

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Unknowns

Defining

$$\mathcal{I} = \{(0,0)\} \cup \{\boldsymbol{k} = (0,k_2) \mid k_2 \neq 0\} \cup \{\boldsymbol{k} = (k_1,k_2) \mid k_1 \neq 0 \text{ and } k_2 \neq 0\},$$

one can identify $x = \{x_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathcal{I}}$.

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Finally, let us define $\mathcal{F} = \{\mathcal{F}_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathcal{I}}$ component-wise by

$$\mathcal{F}_{\boldsymbol{k}} = \begin{cases} \eta, & \boldsymbol{k} = (0,0) \\ g_{\boldsymbol{k}}, & \boldsymbol{k} = (0,k_2), \quad k_2 \neq 0 \\ \begin{pmatrix} f_{\boldsymbol{k}} \\ g_{\boldsymbol{k}} \end{pmatrix}, & \boldsymbol{k} = (k_1,k_2), \quad k_1 \neq 0 \ \text{ and } \ k_2 \neq 0. \end{cases}$$

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**Lemma.** Finding time-periodic solutions $u(t,y)$ of (KS) such that $\eta = 0$ is equivalent to find $x$ such that $\mathcal{F}(x) = 0$.

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**To solve rigorously in a Banach space**

# The Banach space

Define the one-dimensional weights $\omega_k^s$ by

$$\omega_k^s \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } k = 0 \\ |k|^s, & \text{if } k \neq 0. \end{cases}$$

Using the 1-d weights, define the 2-dimensional weights, given $\boldsymbol{k} = (k_1, k_2) \in \mathbb{Z}^2$,

$$\omega_{\boldsymbol{k}}^{\boldsymbol{s}} \stackrel{\text{def}}{=} \omega_{k_1}^{s_1} \omega_{k_2}^{s_2}.$$

They are used to define the norm

$$\|x\|_{\boldsymbol{s}} = \sup_{\boldsymbol{k} \in \mathcal{I}} \omega_{\boldsymbol{k}}^{\boldsymbol{s}} |x_{\boldsymbol{k}}|_\infty,$$

where $|x_{\boldsymbol{k}}|_\infty$ is the sup norm of the vector $x_{\boldsymbol{k}}$, which is one or two dimensional, depending on $\boldsymbol{k}$. Define the Banach space

$$X^{\boldsymbol{s}} = \left\{ x \mid \|x\|_{\boldsymbol{s}} < \infty \right\},$$

consisting of sequences with algebraically decaying tails according to the rate $\boldsymbol{s}$.

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They are used to define the norm

$$\|x\|_{\boldsymbol{s}} = \sup_{\boldsymbol{k} \in \mathcal{I}} \omega_{\boldsymbol{k}}^{\boldsymbol{s}} |x_{\boldsymbol{k}}|_\infty,$$

where $|x_{\boldsymbol{k}}|_\infty$ is the sup norm of the vector $x_{\boldsymbol{k}}$, which is one or two dimensional, depending on $\boldsymbol{k}$. Define the Banach space

$$X^{\boldsymbol{s}} = \{x \mid \|x\|_{\boldsymbol{s}} < \infty\},$$

**Banach algebra under discrete convolution**

consisting of sequences with algebraically decaying tails according to the rate $\boldsymbol{s}$.

For sake of simplicity of the presentation, for $\boldsymbol{k} = (k_1, k_2)$ with $k_1 \neq 0$ or $k_2 \neq 0$, let

$$R_{\boldsymbol{k}}(\nu, L) \stackrel{\text{def}}{=} \begin{pmatrix} \nu k_2^4 - k_2^2 & -k_1 L \\ k_1 L & \nu k_2^4 - k_2^2 \end{pmatrix} \quad \text{and} \quad R_{0,k_2}(\nu, L) \stackrel{\text{def}}{=} \nu k_2^4 - k_2^2,$$

$$\mathcal{N}_{\boldsymbol{k}}(x) \stackrel{\text{def}}{=} \sum_{\boldsymbol{k}^1 + \boldsymbol{k}^2 = \boldsymbol{k}} \begin{pmatrix} 2a_{\boldsymbol{k}^1} b_{\boldsymbol{k}^2} \\ -a_{\boldsymbol{k}^1} a_{\boldsymbol{k}^2} + b_{\boldsymbol{k}^1} b_{\boldsymbol{k}^2} \end{pmatrix}$$

so that one has that

$$\mathcal{F}_{\boldsymbol{k}}(x, \nu) = R_{\boldsymbol{k}}(\nu, L) x_{\boldsymbol{k}} + k_2 \mathcal{N}_{\boldsymbol{k}}(x).$$

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$$\mathcal{F}_{\boldsymbol{k}}(x, \nu) = R_{\boldsymbol{k}}(\nu, L) x_{\boldsymbol{k}} + k_2 \mathcal{N}_{\boldsymbol{k}}(x).$$

**Lemma. (Bootstrap)** Consider a fixed decay rate $\boldsymbol{s} > (1,1)$ and assume the existence of $\boldsymbol{M} > (0,0)$ such that $R_{\boldsymbol{k}}(\nu, L)$ is invertible for all $|\boldsymbol{k}| > \boldsymbol{M}$. If there exists $x \in X^{\boldsymbol{s}}$ such that $\mathcal{F}(x) = 0$, then $x \in X^{\boldsymbol{s}_0}$, for all $\boldsymbol{s}_0 > (1,1)$.

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**Hence, we focus our attention on looking for zeros of F within a Banach space with a fixed decay rate s>(1,1).**

Given $\boldsymbol{m} = (m_1, m_2)$, define $\boldsymbol{F_m} = F_{m_1} \times F_{m_2}$, where $F_{m_j} \stackrel{\text{def}}{=} \{k_j \in \mathbb{Z} \mid |k_j| < m_j\}$. Consider a *Galerkin projection* of $\mathcal{F}$ of dimension $n = n(\boldsymbol{m}) \stackrel{\text{def}}{=} 2m_1 m_2 - 2m_1 - m_2 + 2$ given by $\mathcal{F}^{(\boldsymbol{m})} \stackrel{\text{def}}{=} \{\mathcal{F}_{\boldsymbol{k}}^{(\boldsymbol{m})}\}_{\boldsymbol{k} \in \boldsymbol{F_m}}$, where $\mathcal{F}^{(\boldsymbol{m})} \colon \mathbb{R}^n \to \mathbb{R}^n$, is given component-wise by

$$\mathcal{F}_{\boldsymbol{k}}^{(\boldsymbol{m})}(x_{\boldsymbol{F_m}}) \stackrel{\text{def}}{=} \mathcal{F}_{\boldsymbol{k}}(x_{\boldsymbol{F_m}}, 0_{\boldsymbol{I_m}}), \qquad \boldsymbol{k} \in \boldsymbol{F_m}.$$

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Consider $\hat{x}_{\boldsymbol{F_m}}$ such that $\mathcal{F}^{(\boldsymbol{m})}(\hat{x}_{\boldsymbol{F_m}}) \approx 0$. Let $\hat{x} \stackrel{\text{def}}{=} (\hat{x}_{\boldsymbol{F_m}}, 0_{\boldsymbol{I_m}}) \in X^{\boldsymbol{s}}$. Assume that the Jacobian matrix $D\mathcal{F}^{(\boldsymbol{m})}(\hat{x}_{\boldsymbol{F_m}})$ is non-singular and let $A_{\boldsymbol{m}}$ an approximation for its inverse.

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Define the action of the linear operator $A$ on $x = \{x_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathcal{I}}$ component-wise by

$$\Big[A(x)\Big]_{\boldsymbol{k}} \stackrel{\text{def}}{=} \begin{cases} \Big[A_{\boldsymbol{m}}(x_{\boldsymbol{F_m}})\Big]_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \in \boldsymbol{F_m} \\ R_{\boldsymbol{k}}(\nu, \hat{L})^{-1} x_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \notin \boldsymbol{F_m}. \end{cases}$$

$$T(x) \stackrel{\text{def}}{=} x - A\mathcal{F}(x).$$

Given $\boldsymbol{m} = (m_1, m_2)$, define $\boldsymbol{F_m} = F_{m_1} \times F_{m_2}$, where $F_{m_j} \stackrel{\text{def}}{=} \{k_j \in \mathbb{Z} \mid |k_j| < m_j\}$. Consider a *Galerkin projection* of $\mathcal{F}$ of dimension $n = n(\boldsymbol{m}) \stackrel{\text{def}}{=} 2m_1m_2 - 2m_1 - m_2 + 2$ given by $\mathcal{F}^{(\boldsymbol{m})} \stackrel{\text{def}}{=} \{\mathcal{F}^{(\boldsymbol{m})}_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \boldsymbol{F_m}}$, where $\mathcal{F}^{(\boldsymbol{m})} \colon \mathbb{R}^n \to \mathbb{R}^n$, is given component-wise by

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**(Newton-like operator)**

Given $\boldsymbol{m} = (m_1, m_2)$, define $\boldsymbol{F_m} = F_{m_1} \times F_{m_2}$, where $F_{m_j} \stackrel{\text{def}}{=} \{k_j \in \mathbb{Z} \mid |k_j| < m_j\}$. Consider a *Galerkin projection* of $\mathcal{F}$ of dimension $n = n(\boldsymbol{m}) \stackrel{\text{def}}{=} 2m_1m_2 - 2m_1 - m_2 + 2$ given by $\mathcal{F}^{(\boldsymbol{m})} \stackrel{\text{def}}{=} \{\mathcal{F}^{(\boldsymbol{m})}_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\boldsymbol{F_m}}$, where $\mathcal{F}^{(\boldsymbol{m})} \colon \mathbb{R}^n \to \mathbb{R}^n$, is given component-wise by

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**(Newton-like operator)**

**Lemma.** Consider a Galerkin projection dimension $\boldsymbol{m} = (m_1, m_2)$ and let $\boldsymbol{s} = (s_1, s_2) > (1, 1)$ a decay rate. The solutions of $\mathcal{F} = 0$ are in one to one correspondence with the fixed points of $T$. Also, one has that $T : X^{\boldsymbol{s}} \to X^{\boldsymbol{s}}$.

The rigorous continuation method is based on the notion of the radii polynomials, which provide a numerically efficient way to verify that the operator $T$ is a contraction on a small closed ball $B(\hat{x}, r)$ centered at the numerical approximation $\hat{x}$ in $X^{\boldsymbol{s}}$.

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**Ingredients to construct the radii polynomials**

- Convolution estimates
- Interval arithmetic
- Fast Fourier transform

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## Ingredients to construct the radii polynomials

- Convolution estimates
- Interval arithmetic
- Fast Fourier transform

The closed ball of radius $r$ in $X^{\boldsymbol{s}}$, centered at the origin, is given by

$$B(r) \stackrel{\text{def}}{=} \prod_{\boldsymbol{k}\in\mathcal{I}} \left[-\frac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}, \frac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}\right]^{d(\boldsymbol{k})},$$

where $d(\boldsymbol{k}) = 1$ if $\boldsymbol{k} = (0, k_2)$ and $d(\boldsymbol{k}) = 2$ otherwise. The closed ball of radius $r$ centered at $\hat{x}$ is then

$$B(\hat{x},r) \stackrel{\text{def}}{=} \hat{x} + B(r).$$

Consider now bounds $Y_{\boldsymbol{k}}$ and $Z_{\boldsymbol{k}}$ for all $\boldsymbol{k} \in \mathcal{I}$, such that

$$\left| \left[ T(\hat{x}) - \hat{x} \right]_{\boldsymbol{k}} \right| \leq Y_{\boldsymbol{k}},$$

and

$$\sup_{x_1, x_2 \in B(r)} \left| \left[ DT(\hat{x} + x_1) x_2 \right]_{\boldsymbol{k}} \right| \leq Z_{\boldsymbol{k}}(r).$$

**Lemma.** If there exists an $r > 0$ such that $\|Y + Z\|_{\boldsymbol{s}} < r$, with $Y \stackrel{\text{def}}{=} \{Y_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathcal{I}}$ and $Z \stackrel{\text{def}}{=} \{Z_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathcal{I}}$, then $T$ is a contraction mapping on $B(\hat{x}, r)$ with contraction constant at most $\|Y + Z\|_{\boldsymbol{s}}/r < 1$. Furthermore, there is a unique $\tilde{x} \in B(\hat{x}, r)$ such that $\mathcal{F}(\tilde{x}) = 0$.

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Define the *finite radii polynomials* $\{p_{\boldsymbol{k}}(r)\}_{\boldsymbol{k} \in \boldsymbol{F_M}}$ by

$$p_{\boldsymbol{k}}(r) \stackrel{\text{def}}{=} Y_{\boldsymbol{k}} + Z_{\boldsymbol{k}}(r) - \frac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \mathbb{I}^{d(\boldsymbol{k})},$$

and the *tail radii polynomial* by

$$\tilde{p}_{\boldsymbol{M}}(r) \stackrel{\text{def}}{=} \tilde{Z}_{\boldsymbol{M}}(r) - 1.$$

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**asymptotic bound for $Z_k$ in $X^s$**

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**Lemma.** If there exists an $r > 0$ such that $\|Y + Z\|_{\boldsymbol{s}} < r$, with $Y \stackrel{\text{def}}{=} \{Y_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathcal{I}}$ and $Z \stackrel{\text{def}}{=} \{Z_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathcal{I}}$, then $T$ is a contraction mapping on $B(\hat{x}, r)$ with contraction constant at most $\|Y + Z\|_{\boldsymbol{s}}/r < 1$. Furthermore, there is a unique $\tilde{x} \in B(\hat{x}, r)$ such that $\mathcal{F}(\tilde{x}) = 0$.

Define the *finite radii polynomials* $\{p_{\boldsymbol{k}}(r)\}_{\boldsymbol{k} \in \boldsymbol{F_M}}$ by

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**asymptotic bound for $\mathbf{Z_k}$ in $\mathbf{X^s}$**

$$\tilde{p}_{\boldsymbol{M}}(r) \stackrel{\text{def}}{=} \tilde{Z}_{\boldsymbol{M}}(r) - 1.$$

**Lemma.** If there exists $r > 0$ such that $p_{\boldsymbol{k}}(r) < 0$ for all $\boldsymbol{k} \in \boldsymbol{F_M}$ and $\tilde{p}_{\boldsymbol{M}}(r) < 0$, then there is a unique $\tilde{x} \in B(\hat{x}, r)$ such that $\mathcal{F}(\tilde{x}) = 0$.

# Results

## Kuramoto-Sivashinski equation

$$
(\mathrm{KS}) \left\{ \begin{array}{l} u_t = -\nu u_{yyyy} - u_{yy} + 2uu_y \\ u(t,y) = u(t,y+2\pi), \quad u(t,-y) = -u(t,y) \end{array} \right.
$$

# Results

## Kuramoto-Sivashinski equation

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$$
\boldsymbol{m} = (77, 15), \; \boldsymbol{M} = (229, 43), \; \boldsymbol{s} = (\tfrac{3}{2}, \tfrac{3}{2})
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# of time Fourier modes

# of space Fourier modes

decay rates

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$$\nu \in \{.127, .12707, .12715, .12725, .12739, .12756, .12777\}$$

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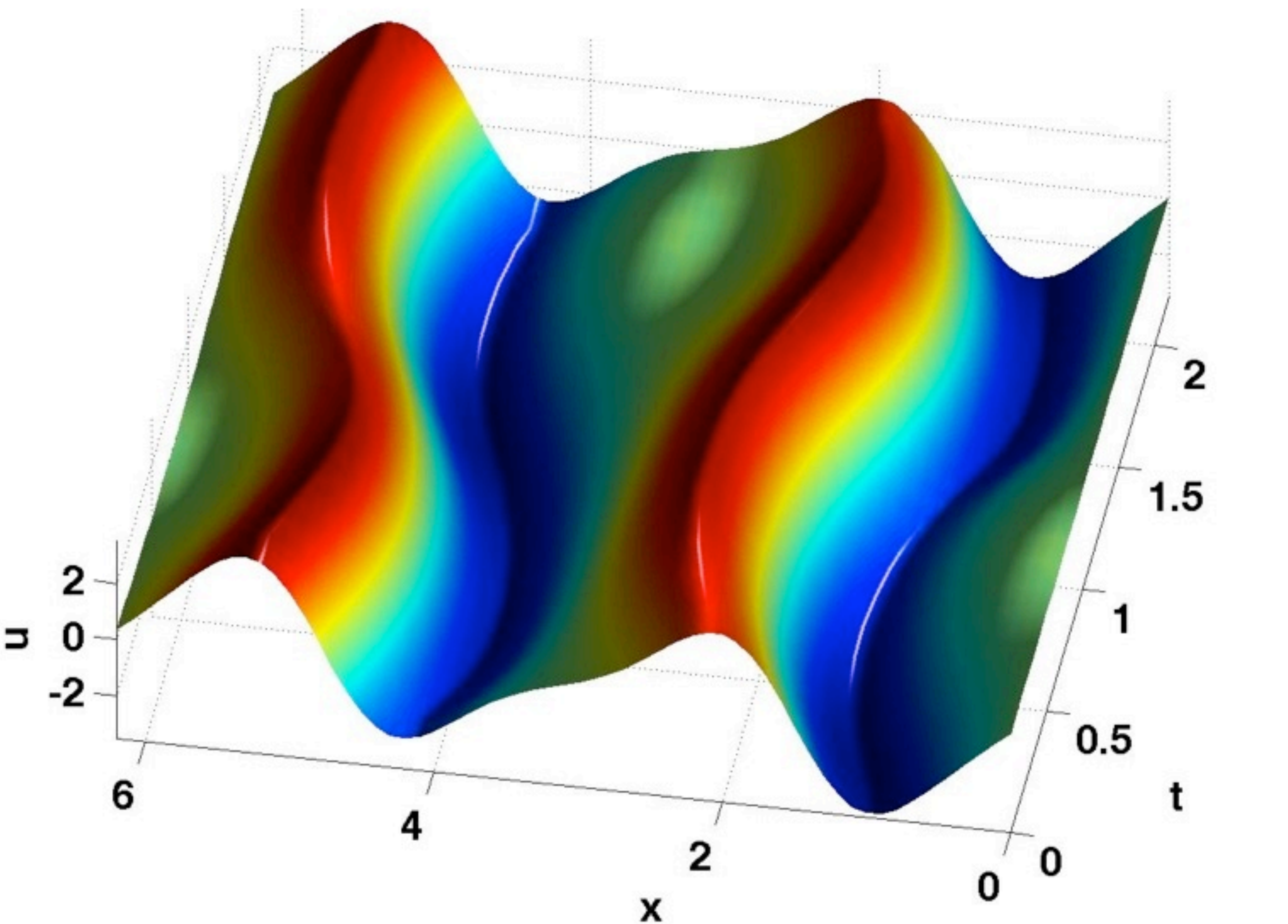
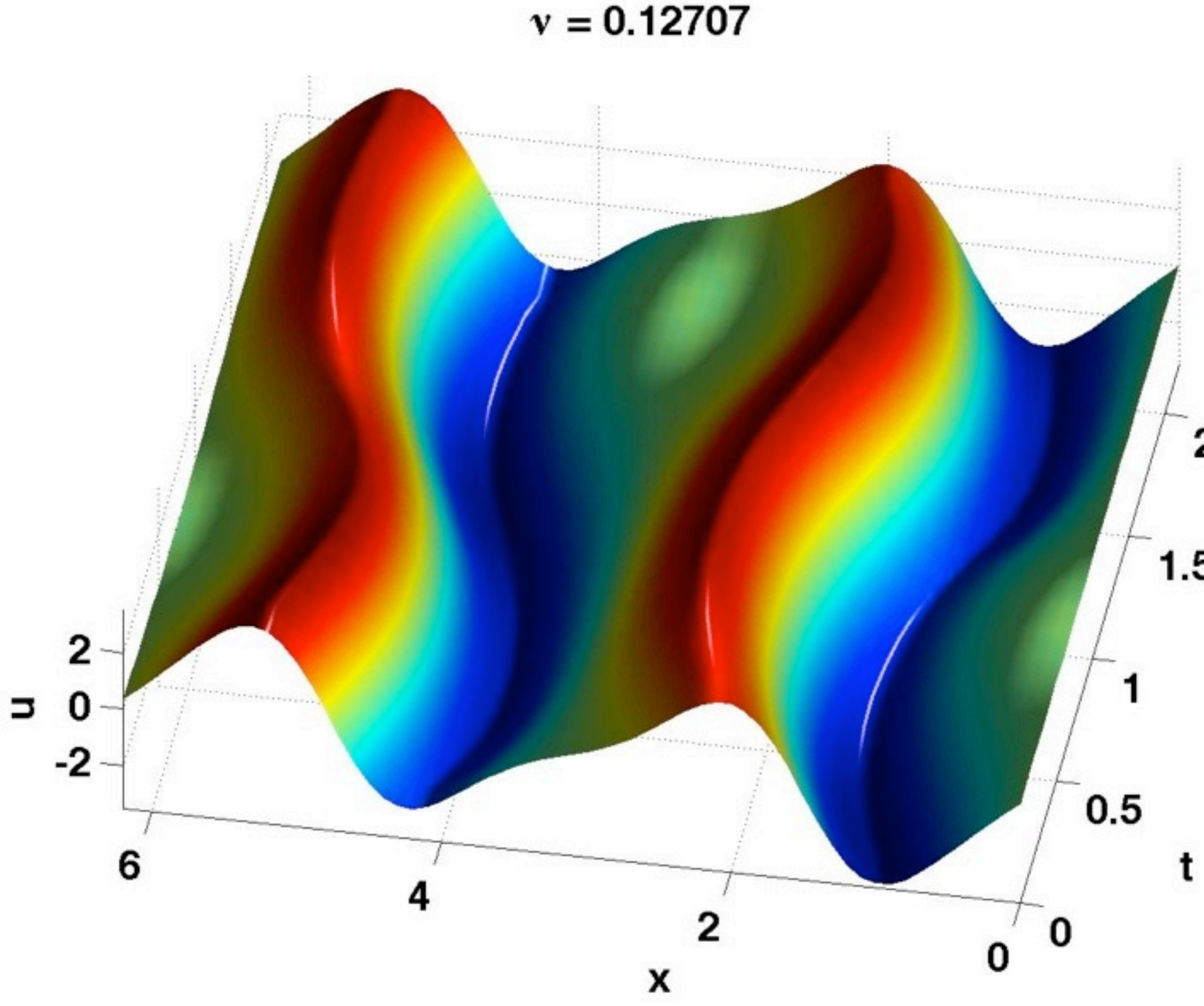
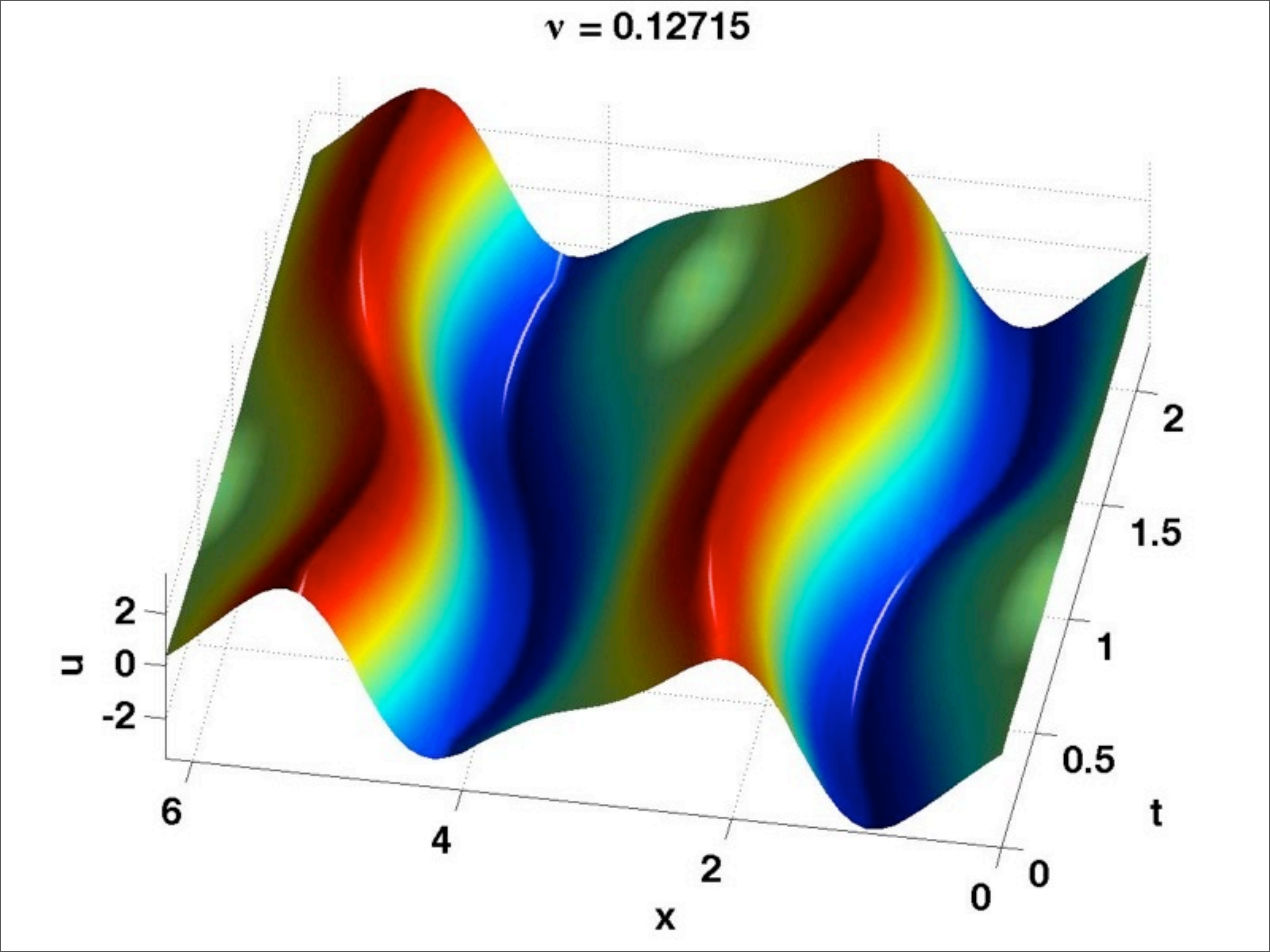
$$\nu \in \{.127, .12707, .12715, .12725, .12739, .12756, .12777\}$$

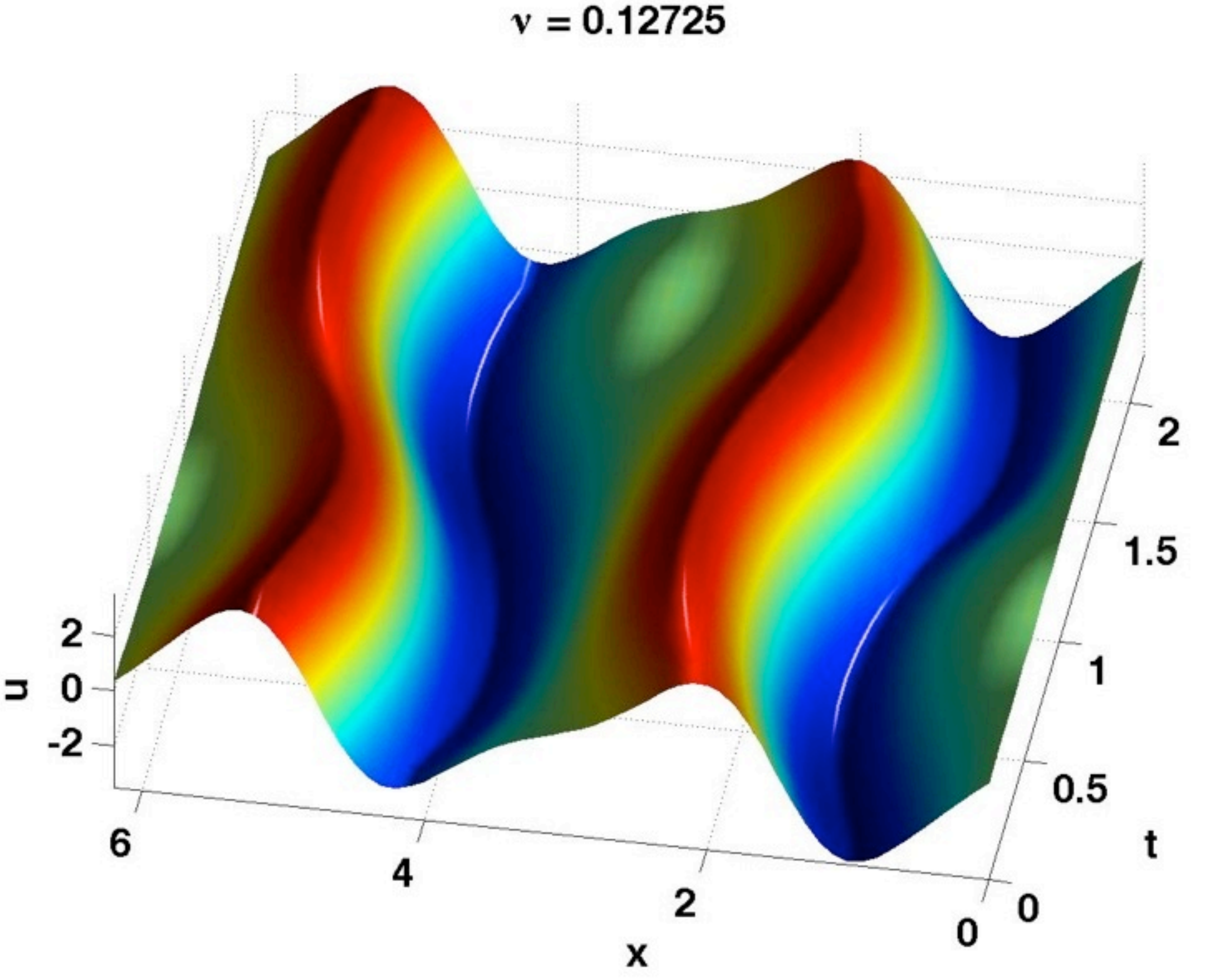
$$\tilde{x} \in B(\hat{x}, r) = \hat{x} + \prod_{\boldsymbol{k} \in \mathcal{I}} \left[ -\frac{3 \times 10^{-4}}{k_1^{3/2} k_2^{3/2}}, \frac{3 \times 10^{-4}}{k_1^{3/2} k_2^{3/2}} \right]^{d(\boldsymbol{k})} \subset X^{\left(\frac{3}{2}, \frac{3}{2}\right)}$$

ν = 0.127

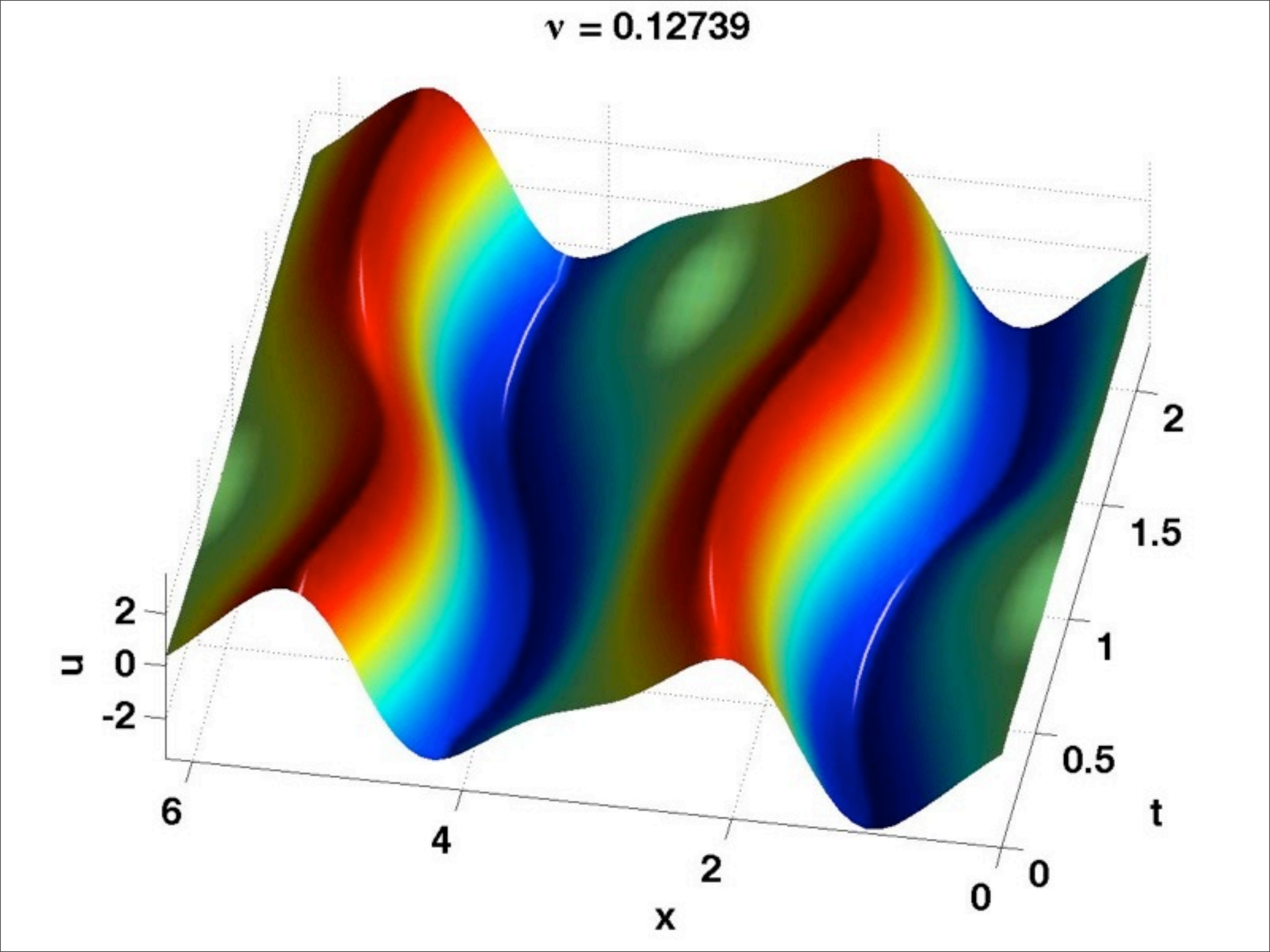


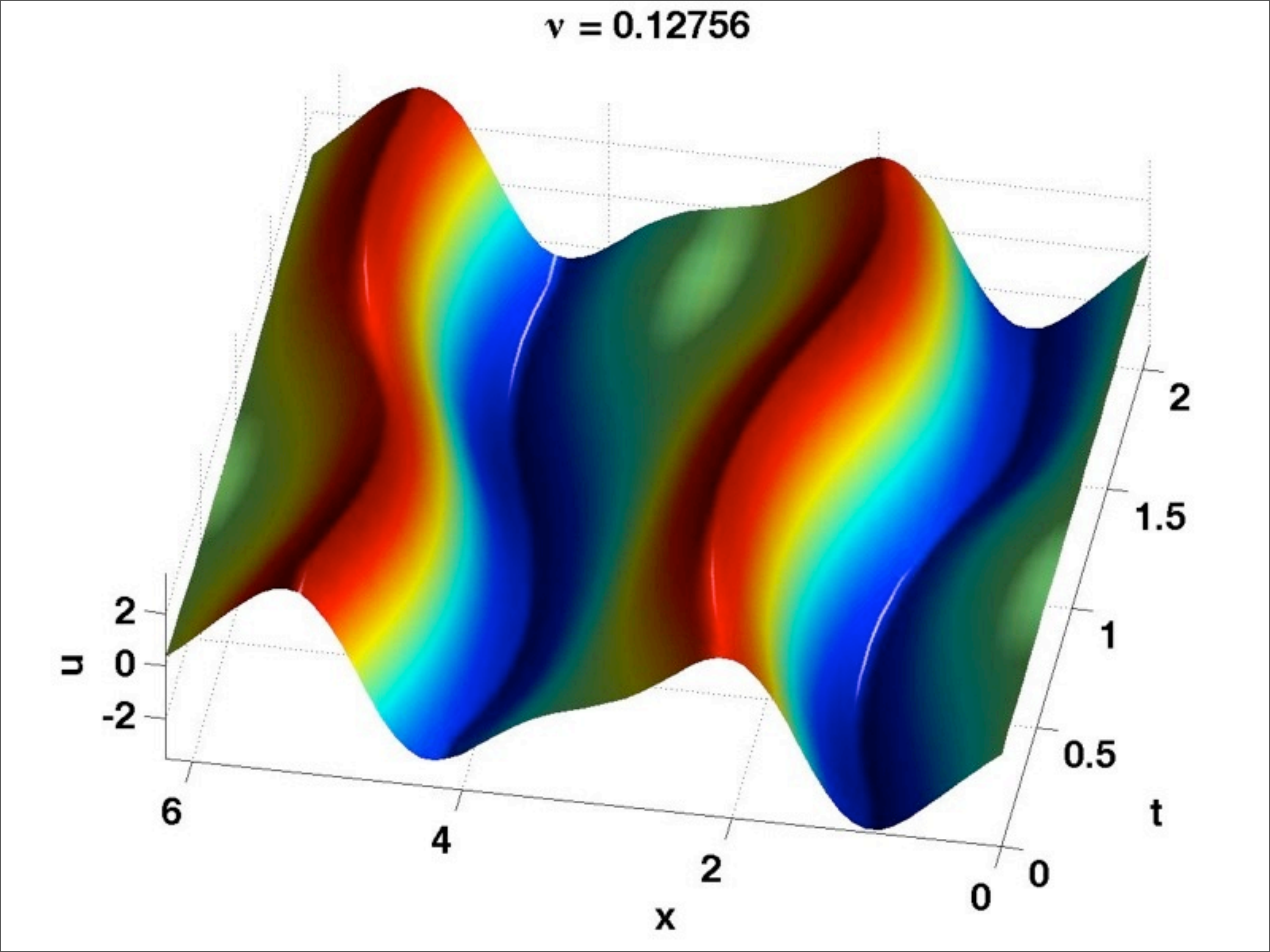

ν = 0.12707
u
2
0
-2
6
4
2
0
x
0
0.5
1
1.5
2
t

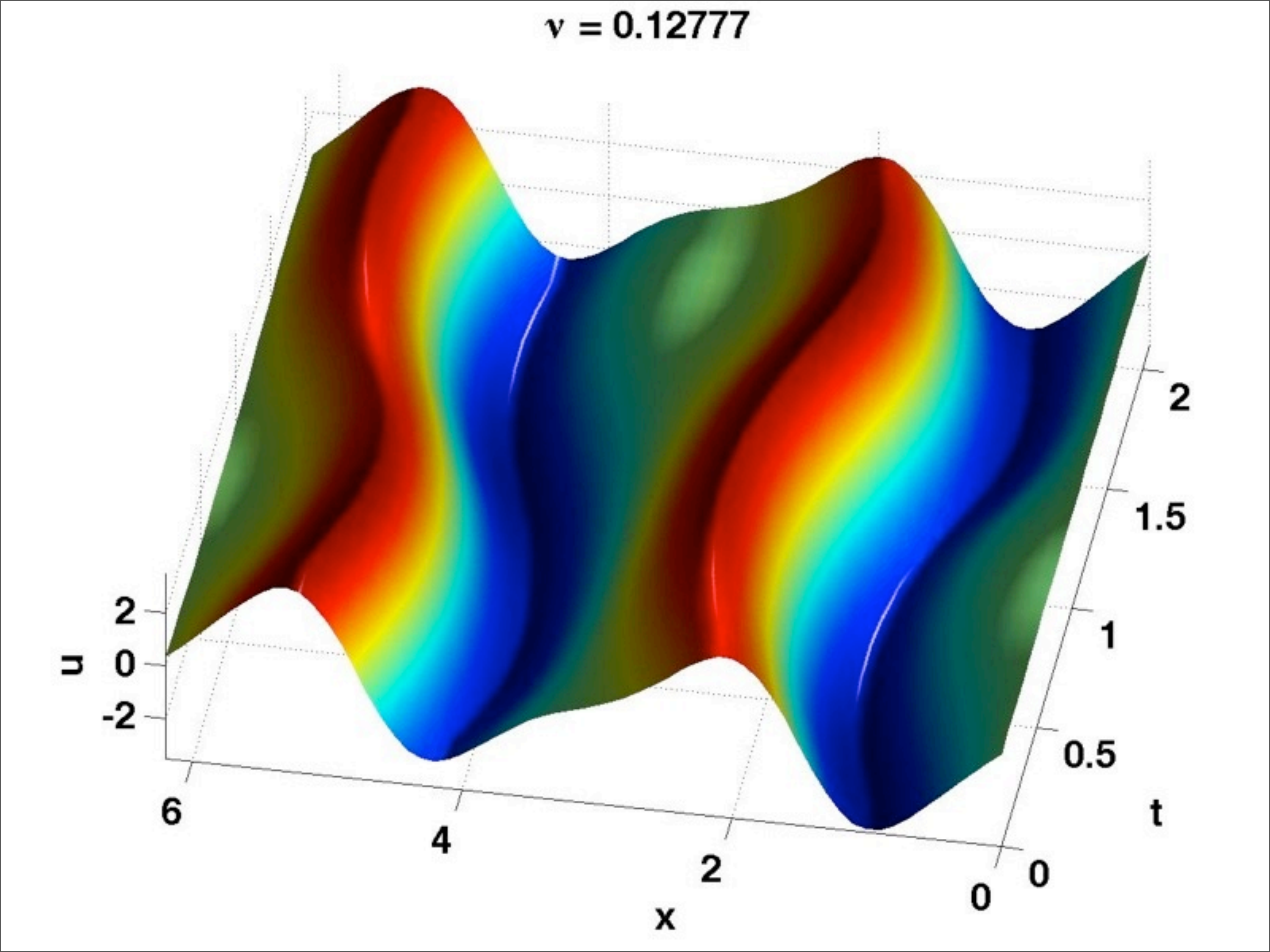

ν = 0.12715
u
2
0
-2
6
4
2
0
x
t
0
0.5
1
1.5
2



ν = 0.12725
u
2
0
-2
6
4
2
0
x
t
0
0.5
1
1.5
2


ν = 0.12739
u
2
0
-2
x
6
4
2
0
t
0
0.5
1
1.5
2


ν = 0.12756
u
2
0
-2
6
4
2
0
x
t
0
0.5
1
1.5
2



ν = 0.12777
u
2
0
-2
6
4
2
0
x
0
0.5
1
1.5
2
t

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