A Transport Theorem for Irregular Evolving Domains

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the Legacy of Jerry Marsden

July 20, 2012

Classical transport theorem

- $(f_t : \mathcal{E} \longrightarrow \mathcal{E} \mid t \in \mathcal{I})$ family of diffeomorphisms — flow
- $\mathcal{M}(t) = f_t(M)$ time-dependent manifold

$$\dot{\overline{\int_{\mathcal{M}} \omega}} = \int_{\mathcal{M}} \dot{\omega} + \int_{\mathcal{M}} i_{\mathbf{v}} d\omega + \int_{\partial\mathcal{M}} i_{\mathbf{v}} \omega$$

- $\mathbf{v}$ velocity of flow

What if the evolution of the domain is **not** given by a flow?

- develop holes
- split into pieces
- boundary could develop corners

Differential chains

differential chain ~ domain of integration

$J \in \hat{\mathcal{B}}_k^r$ = differential k-chains of class r

$\boldsymbol{\omega} \in (\hat{\mathcal{B}}_k^r)'$= differential k-forms of a certain regularity

$$(\boldsymbol{\omega}, J) \longmapsto \int_J \boldsymbol{\omega}$$

Theorem

Let $\mathcal{M}$ be a k-dimensional, compact, Lipschitz submanifold. There is a $J \in \hat{\mathcal{B}}_k^1$ that represents $\mathcal{M}$, in the sense that

$$\int_{\mathcal{M}} \boldsymbol{\omega} = \int_J \boldsymbol{\omega} \quad \text{for all} \quad \boldsymbol{\omega} \in (\hat{\mathcal{B}}_k^1)'.$$

Constructing the space

1. start simple

$$(q;\boldsymbol{\alpha}) \qquad q \text{ point, } \boldsymbol{\alpha} \text{ simple skew } k\text{-form}$$

$$\int_{(q;\boldsymbol{\alpha})} \omega := \omega(q)\cdot\boldsymbol{\alpha}$$

2. linear combinations

$$A = \sum_{i\in I}(q_i;\boldsymbol{\alpha}_i) \qquad \int_A \omega := \sum_{i\in I}\omega(q_i)\cdot\boldsymbol{\alpha}_i$$

3. introduce the B^r norm "the magic"
4. take limits

$$A_m = \sum_{i_m\in I_m}(q_{i_m};\boldsymbol{\alpha}_{i_m}) \xrightarrow{B^r} J \in \hat{\mathcal{B}}_k^r$$

$$\int_J \omega = \lim_{m\to\infty}\sum_{i_m\in I_m}\omega(q_{i_m})\cdot\boldsymbol{\alpha}_{i_m} \qquad \textbf{Riemann sums!}$$

Operators on chains

Boundary

The boundary $\partial \in \mathrm{Lin}(\hat{\mathcal{B}}^r_k, \hat{\mathcal{B}}^{r+1}_{k-1})$ exists and

$$\int_{\partial J} \omega = \int_J d\omega \quad \text{for all} \quad J \in \hat{\mathcal{B}}^r_k, \ \omega \in (\hat{\mathcal{B}}^{r+1}_{k-1})'.$$

$\partial^* = d$ **A statement about adjoints!**

A space of evolving domains

1. start simple

$$(q(t);\boldsymbol{\alpha}(t)) \qquad t \in \mathcal{I}$$

2. linear combinations

$$A(t) = \sum_{i \in I} (q_i(t);\boldsymbol{\alpha}_i(t))$$

3. introduce the C^1_r norm
4. take limits $A_m \xrightarrow{C^1_r} J \in \hat{\mathcal{B}}^r_k[\mathcal{I}]$

$J : \mathcal{I} \longrightarrow \hat{\mathcal{B}}^r_k$ **represents an evolving domain**

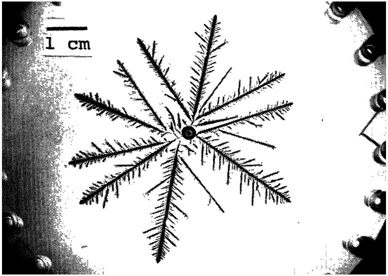
Generalized transport theorem

Let $J \in \hat{\mathcal{B}}_k^r[\mathcal{I}]$ and $\omega \in C^1(\mathcal{I}, (\hat{\mathcal{B}}_k^{r+1})')$ be given. The function $⨍_J \omega$ is differentiable and for all $t \in \mathcal{I}$

$$\left(\overline{⨍_J \omega}^{\,\cdot}\right)(t) = ⨍_{J(t)} \dot{\omega}(t) + ⨍_{E_t J} d\omega(t) + ⨍_{E_t \partial J} \omega(t) \qquad \text{if } k \neq 0.$$

$$\overline{\int_{\mathcal{M}} \omega}^{\,\cdot} = \int_{\mathcal{M}} \dot{\omega} + \int_{\mathcal{M}} i_{\mathbf{v}} d\omega + \int_{\partial \mathcal{M}} i_{\mathbf{v}} \omega$$

Possible applications



- phase transitions
- calculus of variations
- fracture mechanics
- diffusion
- heat conduction

Thanks!