Geodesic equations on contactomorphism groups

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July 24, 2012

Arnold (1966) observed that the geodesic equation on the group of volume-preserving diffeomorphisms of a space M

$$\mathcal{D}_\mu(M) = \{\eta \colon M \to M \,|\, \mathrm{Jac}(\eta) \equiv 1\}$$

are precisely the equation of ideal fluid mechanics, written in Lagrangian coordinates:

$$\eta_{tt} = -\nabla p \circ \eta, \qquad \mathrm{Jac}(\eta) \equiv 1,$$

equivalent to the Euler equation

$$u_t + u \cdot \nabla u = -\nabla p, \qquad \eta_t = u \circ \eta.$$

The Riemannian metric is just the L^2 inner product (which happens to be right-invariant).

This helps us understand stability (by relating it to curvature), and gives us a sometimes easier way to prove local existence (by using Picard iteration). The latter approach was used by Ebin-Marsden (1970).

Whether this works rigorously depends on whether one can treat the geodesic equation as an ordinary differential equation on an infinite-dimensional manifold.

Typically we do this by enlarging to the group $\mathcal{D}^s_\mu(M)$ of Sobolev H^s diffeomorphisms ($s > \dim(M)/2 + 1$ to ensure C^1), which is a Hilbert manifold, and prove that the geodesic equation is smooth on this manifold.

Usually this all works out as long as the Lagrangian equation doesn't lose derivatives. (Note that the Euler equation always loses derivatives, but this may be canceled out in Lagrangian coordinates.)

If the equation is a genuine ODE, then curvature computations give rigorous results on stability. Otherwise they may be useless.

Other PDEs for which a geodesic interpretation has been found:

- Burgers' equation on $\mathcal{D}(S^1)$;
- Korteweg-de Vries equation on the Virasoro group;
- Camassa-Holm equation on $\mathcal{D}(S^1)$;
- Hunter-Saxton equation on $\mathcal{D}(S^1)/S^1$;
- Magnetohydrodynamics on $\mathcal{D}_\mu(M) \ltimes T_{\mathrm{id}}\mathcal{D}_\mu(M)$;
- Landau-Lifschitz equation on $C^\infty(S^1, SO(3))$;
- Inextensible strings on the space of unit-speed curves,

and several others.

For many of these equations, the approach only works formally; the equation is not a smooth ODE on an infinite-dimensional manifold. (Typically they are hyperbolic PDE; we get solutions with C^0 but not C^1 dependence on initial data.)

The classical diffeomorphism groups

Volumorphisms are one of the three "classical" diffeomorphism groups, with nice algebraic properties. The other two are the diffeomorphisms preserving a symplectic form (on an even-dimensional manifold) or a contact structure (on an odd-dimensional manifold).

- If M has dimension $2n$, a symplectic form ω is a 2-form such that ω^n is nowhere zero. The typical example is $\omega = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n$, and there is a Darboux coordinate chart for any symplectic form which makes it look like this.
- If M has dimension $2n+1$, a contact form θ is a 1-form such that $\theta \wedge (d\theta)^n$ is nowhere zero. The typical example is $\theta = dz + x^1\, dy^1 + \cdots + x^n\, dy^n$. Again there is always a chart to make any contact form look like this. The contact structure is the kernel of the contact form.

Ebin (GAFA, 2011) studied the geodesic equation on the symplectomorphism group. If the symplectic manifold has trivial first cohomology and the metric is compatible with the symplectic form, it takes the form

$$\frac{\partial}{\partial t}\Delta f + u(\Delta f) = 0,$$

where u is the skew-gradient of f, given in Darboux coordinates by

$$u = -\frac{\partial f}{\partial x^1}\frac{\partial}{\partial y^1} + \frac{\partial f}{\partial y^1}\frac{\partial}{\partial x^1} + \cdots - \frac{\partial f}{\partial x^n}\frac{\partial}{\partial y^n} + \frac{\partial f}{\partial y^n}\frac{\partial}{\partial x^n}.$$

This is a genuine ODE on the symplectomorphism group, so one obtains local existence and smooth dependence on initial conditions. Furthermore the fact that Δf is conserved along trajectories implies global existence in the same way as in two-dimensional hydrodynamics.

There are two possible extensions of this result to contact forms.

- The quantomorphism group:

$$\mathcal{D}_q(M) = \{\eta \in \mathcal{D}(M) \,|\, \eta^*\theta = \theta\}$$

- The contactomorphism group

$$\mathcal{D}_\theta(M) = \{\eta \in \mathcal{D}(M) \,|\, \eta^*\theta = \Lambda\theta \text{ for some positive function } \Lambda\}$$

The quantomorphism group may be degenerate depending on properties of the Reeb field. When it is nondegenerate, it is closely related to a symplectomorphism group. Its Lie algebra is isomorphic to functions f such that $E(f) = 0$.

The contactomorphism group consists of diffeomorphisms preserving the contact structure, the kernel of θ, which is often more interesting than a particular contact form. The Lie algebra is isomorphic to the space of all functions, so it is never degenerate.

There is a unique vector field E, called the Reeb field, such that $\theta(E) \equiv 1$ and $\iota_E d\theta = 0$. In $3D$ Darboux coordinates where $\theta = dz + x\,dy$, we have $E = \frac{\partial}{\partial z}$.

The Lie algebra for both contactomorphism groups consists of vector fields on M of the form $u = S_\theta f$, where $f\colon M \to \mathbb{R}$ is a function. The operator S_θ is uniquely specified by the conditions $\theta(u) = f$ and $\iota_u d\theta = E(f)\theta - df$.

In $3D$ Darboux coordinates, we have

$$S_\theta f = \left(x\,\frac{\partial f}{\partial z} - \frac{\partial f}{\partial y}\right)\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\,\frac{\partial}{\partial y} + \left(f - x\,\frac{\partial f}{\partial x}\right)\frac{\partial}{\partial z}.$$

Notice that we differentiate f in only two directions, $\frac{\partial}{\partial x}$ and $x\,\frac{\partial}{\partial z} - \frac{\partial}{\partial y}$!

For the full contactomorphism group, any function f will work. For the quantomorphism group, we must have $E(f) = 0$. Thus if E is not a regular vector field, the quantomorphism group can become degenerate.

For example on $\mathbb{T}^3$, one contact form is $\theta = \sin z\, dx + \cos z\, dy$. Its Reeb field is $E = \sin z\, \frac{\partial}{\partial x} + \cos z\, \frac{\partial}{\partial y}$, for which the orbits are typically dense in the tori $z =$ constant. Hence any such f is constant in both x and y, and the resulting quantomorphism group is abelian. (Thus the geodesic equation is trivial.)

If the Reeb field has all orbits closed and of the same period, there is a quotient manifold N with a symplectic form ω such that the pullback of ω is $d\theta$. In this case the quantomorphism group is a circle bundle over the Hamiltonian diffeomorphism group of N.

Example:

Consider $M = S^3 \subset \mathbb{R}^4$ with θ induced by $-x\,dw + w\,dx - z\,dy + y\,dz$. The Reeb field is just the lift of this 1-form. All orbits are closed and have length 2π.

Use toroidal coordinates on S^3, with $0 < \delta < \frac{\pi}{2}$:

$$w = \sin\delta\cos\beta, \quad x = \sin\delta\sin\beta, \quad y = \cos\delta\cos\gamma, \quad z = \cos\delta\sin\gamma.$$

Then $\theta = \sin^2\delta\,d\beta + \cos^2\delta\,d\gamma$, so that $\theta\wedge d\theta = \sin 2\delta\,d\delta\wedge d\beta\wedge d\gamma$, and the Reeb field is $E = \frac{\partial}{\partial\beta} + \frac{\partial}{\partial\gamma}$. The symplectic quotient is $N = S^2$, and the quotient map is given in standard spherical coordinates (ϕ, ψ) by $\phi = 2\delta$ and $\psi = \beta - \gamma$. This is the Hopf fibration, the prototypical example of a Boothby-Wang fibration.

The induced symplectic form on S^2 is $\omega = 2\sin\phi\,d\phi\wedge d\psi$. The projection is a Riemannian submersion from the unit sphere S^3 to the sphere S^2 with radius 2. This is also true if we replace the metric on S^3 by the Berger metric (with the length of E rescaled).

Quantomorphism geodesics

In the Sobolev H^s context for $s > \dim M/2 + 1$, Ratiu-Schmid (1981) proved that if the Reeb field E is regular (all orbits closed and of the same period), then $\mathcal{D}^s_q(M)$ is a C^∞ Hilbert submanifold of $\mathcal{D}^s_E(M)$, the diffeomorphisms commuting with the flow of the Reeb field. They also showed that the fiber bundle map over $\mathcal{D}^s_{\mathrm{ham}}(N)$ onto the symplectic quotient is C^∞.

Smolentsev (1994) derived the geodesic equation formally and computed the Riemannian curvature.

Gay-Balmaz, Gibbons, Holm, Tronci, and Vizman have recently studied the quantomorphism group physically, in several papers, in the context of the geodesic Vlasov equation. They focused more on physical properties and singular solutions than local existence, and studied a more general model of the form

$$\omega_t + \{G \star \omega, \omega\} = 0$$

where G is an operator.

Theorem (Ebin, P.)

If E is regular, then the quantomorphism group $\mathcal{D}^s_q$ is a C^∞ submanifold of $\mathcal{D}^s_{E,\mu} = \mathcal{D}^s_E \cap \mathcal{D}^s_\mu$, and the orthogonal projection is C^∞. Thus the geodesic equation is a C^∞ ODE on $\mathcal{D}^s_q$, and has unique solutions depending smoothly on the initial condition.

Smoothness of the submanifold is proved with the implicit function theorem for Hilbert spaces, while smoothness of the projection is proved using the same technique as in Ebin-Marsden (1970). The difficulty is in showing that an operator like $u \mapsto \big(P(u \circ \eta^{-1})\big) \circ \eta$ is smooth in η, where P is the projection in the identity tangent space, even though composition is *not* smooth.

The basic trick is that a differential operator like $D_\eta \colon f \mapsto \frac{d}{dx}(f \circ \eta^{-1}) \circ \eta$ is smooth in η since it looks like $(D_\eta f)(x) = f'(x)/\eta'(x)$; in other words, all the compositions cancel out, and multiplications are better behaved.

The geodesic equation is equivalent to the Euler-Arnold equation

$$\partial_t \Delta_\theta f + u \cdot \nabla \Delta_\theta f = 0,$$

where $u = S_\theta f$ and $\Delta_\theta f = S_\theta^* S_\theta f$.

This is an active-scalar equation. Since $E(f) = 0$, we can view it as an equation for a function on the symplectic quotient N. If $M = S^3$ with the Berger metric ($|E_1| = \alpha$ and $|E_2| = |E_3| = 1$), the resulting equation on S^2 is the quasigeostrophic equation

$$\partial_t(\Delta f - \alpha^2 f) + \{f, \Delta f\} = 0.$$

Theorem (Ebin, P.) Solutions exist for all time.

The proof is similar to Kato (1967) and Ebin (2011), and is based on the vorticity conservation $\Delta_\theta f(t, \eta(t,x)) = \Delta_\theta f(0,x)$, which leads to a $C^{1+\alpha}$ estimate on $u = S_\theta f$. All H^s norms of u can then be estimated in terms of the C^1 bound of u, and global existence follows.

The contactomorphism group

Formally, there are several reasonable choices for a geodesic equation on $\mathcal{D}_\theta(M)$. If we simply used the right-invariant metric induced from $\mathcal{D}(M)$, we would obtain the geodesic equation

$$\Delta_\theta f_t + u \cdot \nabla \Delta_\theta f + (n+2)(E \cdot \nabla f)(\Delta_\theta f) = 0.$$

However Δ_θ is a *subelliptic* operator: it does not differentiate in the Reeb direction. This complicates estimates.

A more serious problem is that $\mathcal{D}^s_\theta(M)$ is not a smooth submanifold of $\mathcal{D}^s(M)$. Hence the proof that the geodesic equation is an ODE fails (and in fact the result is false even when $M = S^1$).

To understand this, suppose v is an H^s vector field and $v = S_\theta f$ for some function f. Then $f = \theta(v)$ is also H^s, but v involves derivatives of f, so it is only in H^{s-1}. The problem is that $S_\theta f$ is not a first-order operator since it doesn't differentiate in the Reeb direction.

This can be easily resolved if we instead view $\mathcal{D}^s_\theta(M)$ as a submanifold of the semidirect product $H^s_+(M) \ltimes \mathcal{D}^s(M)$, via

$$\widetilde{\mathcal{D}^s_\theta}(M) = \{(\Lambda, \eta) \,|\, \eta^*\theta = \Lambda\theta\}.$$

Omori (1974) proved the inclusion is smooth.

The Lie algebra is

$$T_{\mathrm{id}}\widetilde{\mathcal{D}^s_\theta}(M) = \{(E \cdot \nabla f, S_\theta f) \,|\, f \in H^{s+1}(M)\}.$$

This saves us since the first term differentiates f in the Reeb direction as well.

A natural right-invariant Riemannian metric is then given by

$$\|(E \cdot \nabla f, S_\theta f)\|^2 = \int_M (E \cdot \nabla f)^2 + |S_\theta f|^2 \, d\mu.$$

This is equivalent to the H^1 metric on functions f.

The geodesic equation then becomes (if E is a Killing field)

$$\partial_t \overline{\Delta_\theta} f + u \cdot \nabla(\overline{\Delta_\theta} f) + (n+2)(E \cdot \nabla f)(\overline{\Delta_\theta} f) = 0,$$

where $\overline{\Delta_\theta} = \Delta_\theta - (E \cdot \nabla)^2$.

Special case: when $n = 0$, $M = S^1$, $\theta = dx$, and $E = \frac{\partial}{\partial x}$, the two geodesic equations become

$$f_t + 3ff_x = 0 \qquad \text{vs.} \qquad f_t - f_{txx} + 3ff_x - 2f_x f_{xx} - ff_{xxx} = 0.$$

The first (Burgers' equation) is *not* an ODE on $\mathcal{D}(S^1)$, but the second one (Camassa-Holm equation) is. (Constantin-Kolev 2002).

Theorem (Ebin, P.)

For $s > \dim(M)/2 + 1$, the geodesic equation on $\widetilde{\mathcal{D}^s_\theta}(M)$ is a smooth ODE, and hence we have unique solutions for short time which depend smoothly on the initial conditions.

Algebraic digression: The usual geometric interpretation of the Camassa-Holm equation is as the geodesic equation on $\mathcal{D}(S^1)$ with right-invariant H^1 metric

$$\langle\langle u, u\rangle\rangle = \int_{S^1} u(x)^2 + u'(x)^2\,dx.$$

(Misiołek 1998, Kouranbaeva 1999).

An alternative interpretation comes from considering $\mathcal{D}(S^1)$ as a subgroup of $\mathcal{D}(S^1) \ltimes C^\infty(S^1, \mathbb{R})$, under the identification $u \mapsto (u, u')$. This is a Lie algebra homomorphism, since the semidirect product has Lie algebra $[(u, f), (v, g)] = (-uv' + vu', vf' - ug')$. This works because of the formula $\partial_x(uv' - vu') = uv'' - vu''$. The H^1 metric on $\mathcal{D}(S^1)$ is then induced as the submanifold metric of the L^2 metric on the semidirect product.

The punch line is that the semidirect product L^2 geometry is a lot simpler. Hence we get a nice curvature formula for Camassa-Holm geometry using submanifold geometry, which is hard to obtain directly (Khesin, Misiołek, Lenells, P., PAMQ 2012?).

Global existence for the contactomorphism equation is still open. For periodic Camassa-Holm, it is known (Constantin-Escher 1998, McKean 1998) that we have global existence if and only if the function $m_0 = f_0 - \partial_x^2 f_0$ is either nonnegative or nonpositive. The proof uses various conservation laws for m.

We can prove some of the same conservation laws for the contactomorphism equation. Let $m = \overline{\Delta_\theta} f$ be the momentum. Then

- $\int_M f \, d\mu =$ constant.
- $\int_M fm \, d\mu =$ constant (energy conservation).
- If m_+ denotes the positive part of m, then $\int_M (m_+)^{(n+1)/(n+2)} \, d\mu =$ constant.
- If $\eta(t)^*\theta = \Lambda(t)\theta$, then

  $$\partial_t \ln \big(\Lambda(t,x)\big) = (n+2)Ef(t, \eta(t,x))$$

  and

  $$\Lambda(t,x)^{n+2} m(t,x) = m_0(x)$$

  (vorticity conservation).

In many ways this equation seems to have more in common with Camassa-Holm than EPDiff.

For periodic Camassa-Holm, it is known (Constantin-Escher, 1998; McKean, 1998) that the solution breaks down in finite time if and only if the sign of m_0 is not constant. We conjecture the same is true for the contactomorphism equation.

By the same techniques as with the quantomorphism equation, we get global existence if we have a C^0 bound on $m(t)$. Vorticity conservation implies that a C^0 bound on $Ef(t)$ is sufficient. Hence blowup should be a one-dimensional phenomenon.

There are also peakon solutions: solving $\Delta_\theta f = \delta$ and translating by the Reeb flow gives the same kind of soliton solution as in the Camassa-Holm equation. The interactions here are completely unknown.

Integrability is also unknown. Complete integrability seems unlikely since the symplectomorphism equation is not integrable, but there may still be infinitely many conservation laws.

This is the end of the talk. Thanks!