Fingering instability down the outside of a vertical cylinder

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joint work with

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Workshop on Surfactant Driven Thin Film Flows

The Fields Institute

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Thin Fluid Layers Spreading on Solid Substrates

Applications to Coating Processes

Biological: Lungs, Tear films.

Manufacturing: Applying paints, fabricating semi-conductors, coating medications.

Geological: Lava Flows.

A moving contact line of a thin film can undergo a fingering instability in the presence of external forcing such as gravitational, centrifugal or Marangoni.

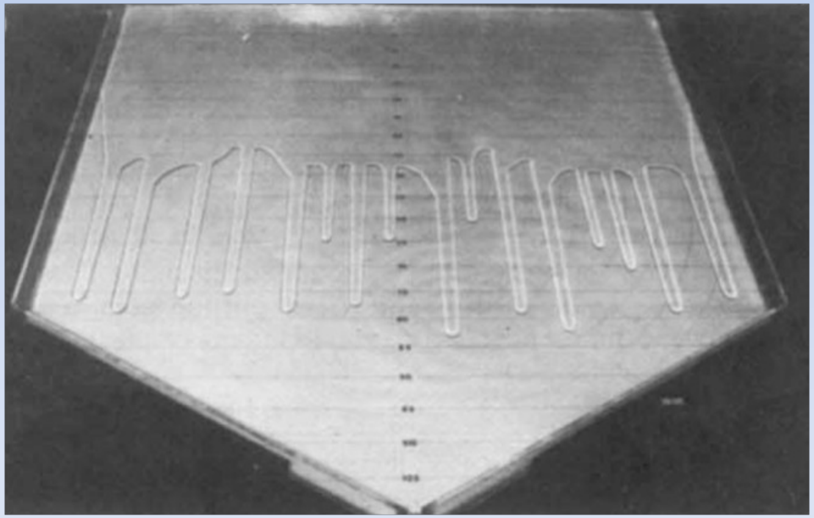


Image from Huppert, 1982.

Background on Gravity-Driven Contact Lines

Inclined and Vertical Planes

- **Experimental:** Huppert, *Nature* 1982; Silvi & Dussan, *Phys. Fluids* 1985; de Bruyn, *Phys. Rev A* 1992, Jerrett & de Bruyn, *Phys. Fluids A* 1992, Kondic, *SIAM Review* 2003.

- **Theoretical:** Troian, Safran, Herbolzheimer, Joanny, *Europhys. Lett.* 1989; Spaid & Homsy, *Phys. Fluids* 1996; Bertozzi & Brenner, *Phys. Fluids* 1997.

- **Review Article on Thin Films:** Craster & Matar, *Rev. Mod. Phys.* 2009.

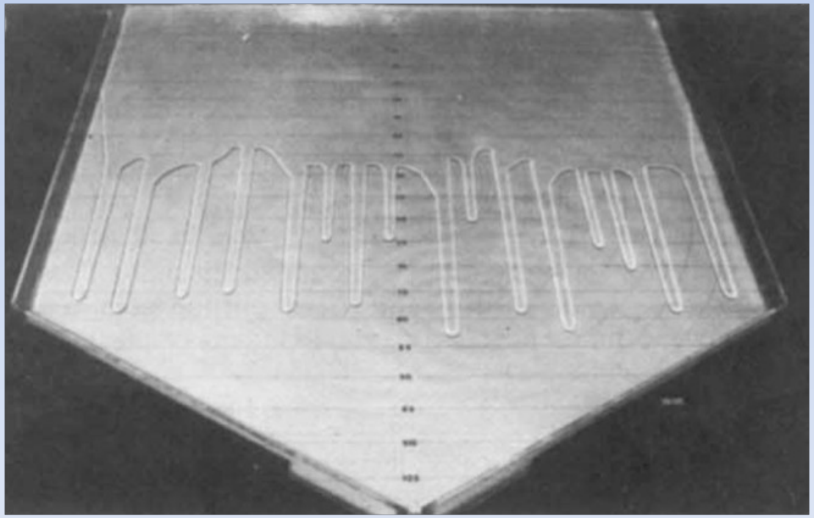


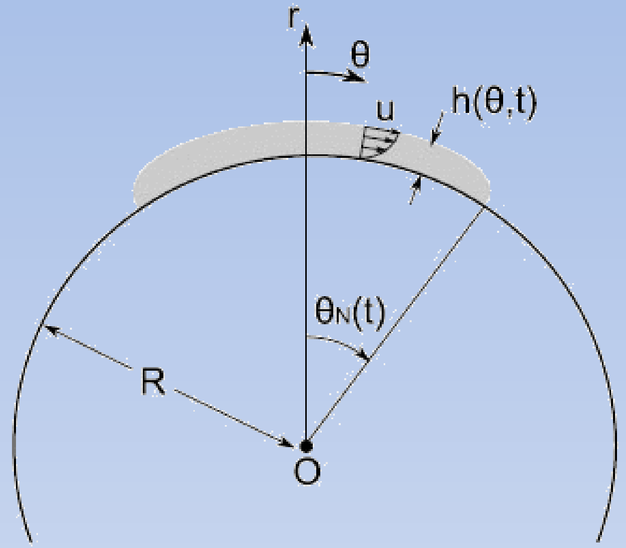
Image from Huppert, 1982.

Background on Gravity-Driven Contact Lines

Horizontal Cylinder and Sphere

- **Outside Horizontal Cylinder and Sphere:**

  Takagi and Huppert, *JFM* 2010.



Coating a horizontal cylinder

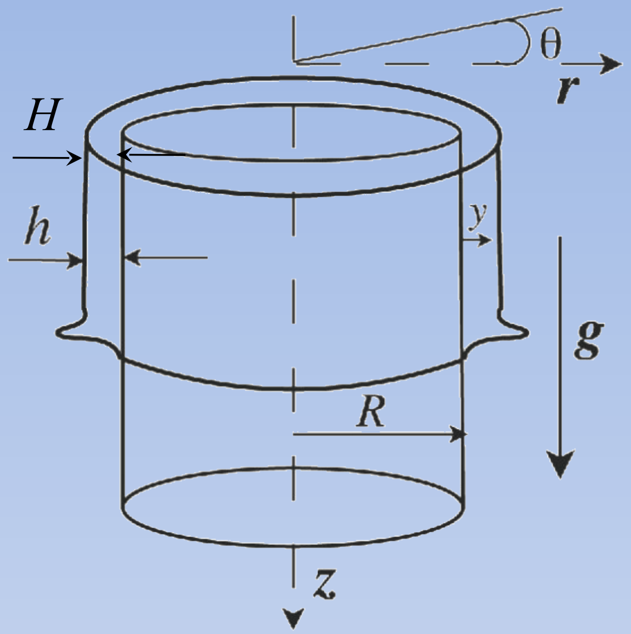
Axisymmetric flow down a sphere

Images from Takagi and Huppert

Our Study on Gravity-Driven Contact Lines

Vertical Cylinder

Consider the dynamics of an axisymmetric contact line.



Azimuthal curvature distinguishes a cylindrical substrate from a planar one. Understanding the effects of curvature is useful for coating cylindrical surfaces from wires to conduits.

Questions

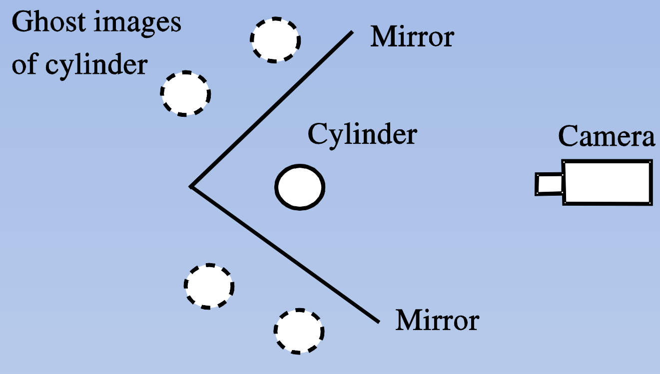
•Does substrate curvature effect features of the contact line?

•Is there a critical radius **above** which substrate curvature (= 1/R) is negligible and the contact line dynamics match those on a vertical plane?

•Is there a critical radius **below** which fingering is inhibited?

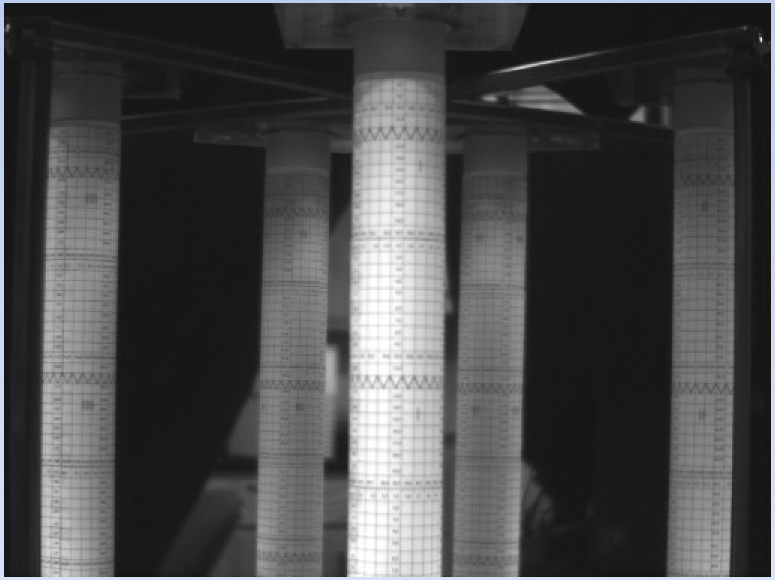
Experimental Details

Top View of Experiment

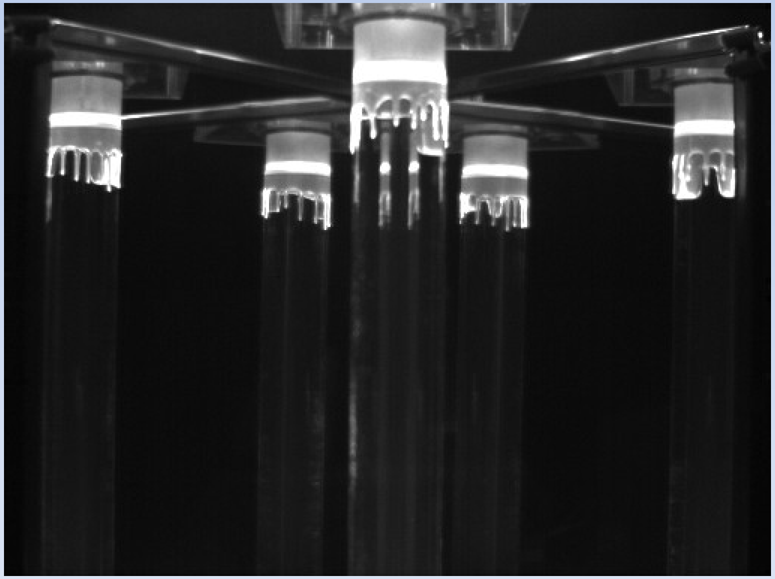


Six clear acrylic cylinders (61 cm tall): **R = 0.159 - 3.81 cm.**
Fluids: Glycerin & Silicone oil (1000 cSt).
Flourescent dye and black lights illuminate fluid.
Mirrors used to visualize around the cylinder periphery.
Fluid delivery: Reservoir cup atop cylinder or fluid pump.
Gap height between fluid source and cylinder is 0.105 cm.
Framing Rate: 100 frames/s.
Image size: 512 x 384 pixels; Pixel resolution: 6 pixels/cm.

Calibration Image



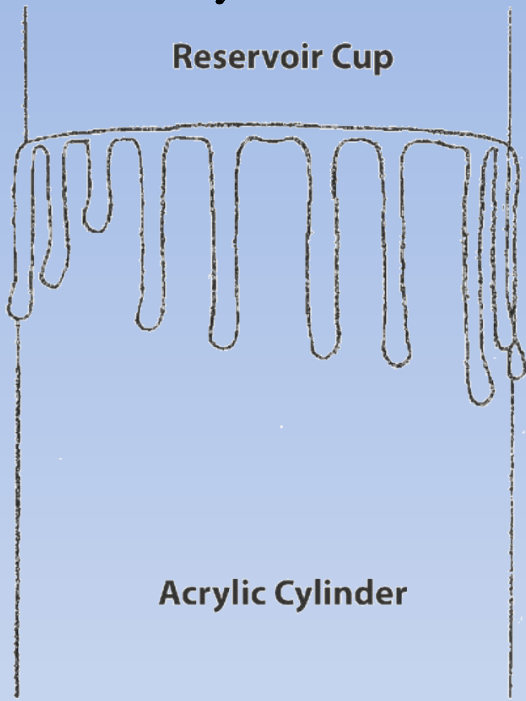
Silicone oil - R = 3.81 cm



The contact line develops a fingering pattern in all of the experiments.

Experiments: Fingering Pattern

Glycerin

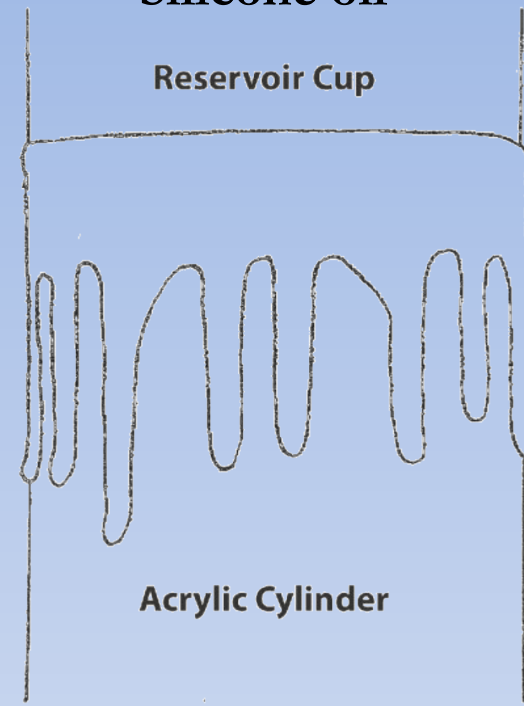


Glycerin
γ = 58.4 dyn/cm
ν = 8.0 cm^2/s

Silicone Oil
γ = 21.9 dyn/cm
ν = 10.3 cm^2/s

γ = surface tension
ν = kinematic viscosity
R = 3.81 cm

Silicone oil



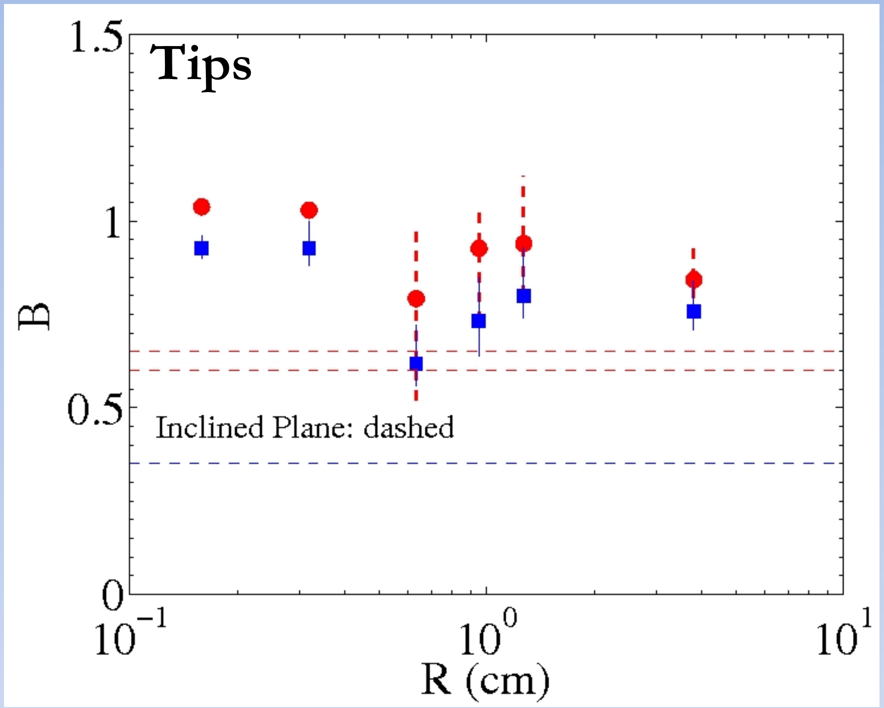
- Glycerin **partially wets** cylinder. Fingers form long straight rivulets with stationary troughs; same as behavior down an inclined plane*.

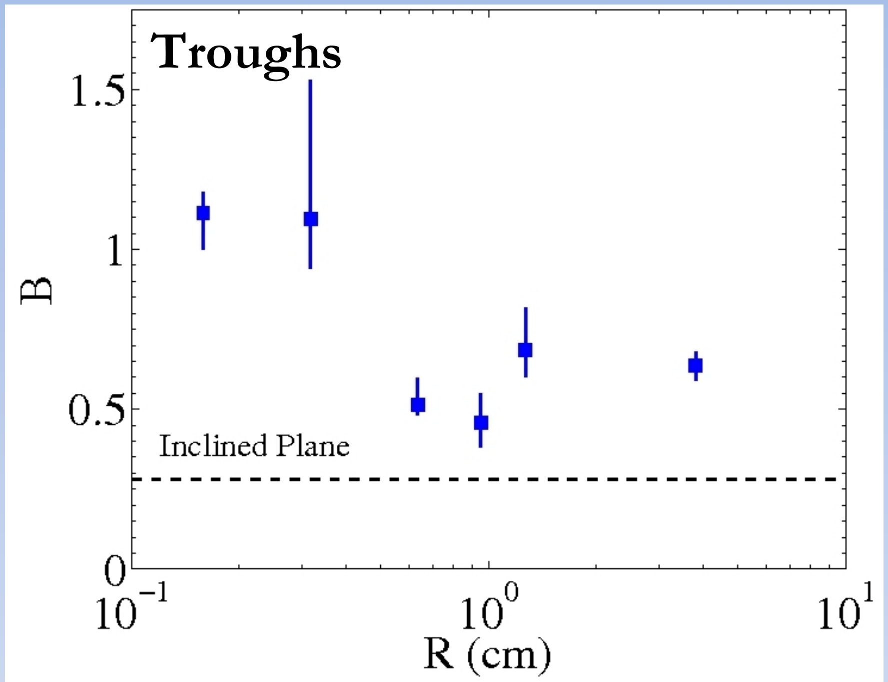
- Silicone oil **completely wets** as tips and troughs travel down the cylinder. **Effect of substrate curvature:** Fingers do not form a **sawtooth** pattern as in inclined plane experiments*.

* Huppert, *Nature* 1982; Silvi & Dussan, *Phys. Fluids* 1985; de Bruyn, *Phys. Rev A* 1992

Experiments: Finger Motion

Position of tips and troughs follow the power-law scaling $z = A(t - t_0)^B$



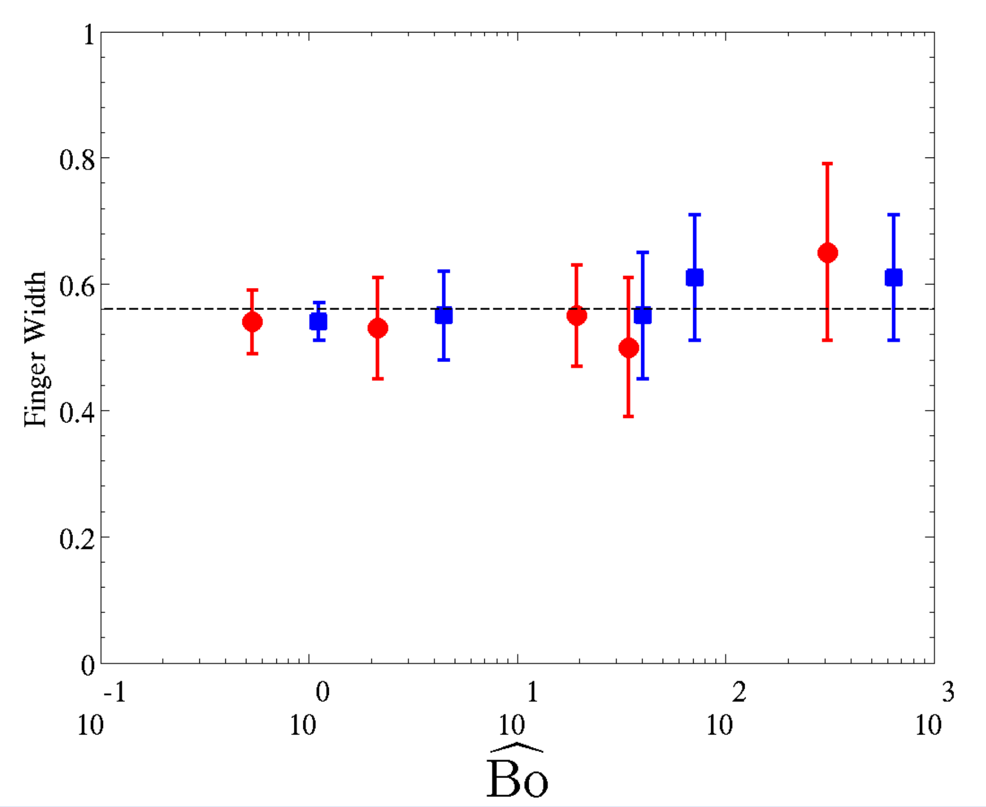


- **Glycerin** tips travel faster than **silicone oil** tips.
- Tips and silicone oil troughs travel faster down vertical cylinder than an inclined plane.*
- Data does not scale with cylinder radius.

Glycerin: Red
Silicone Oil: Blue

* Huppert, *Nature* 1982; Jerrett & de Bruyn, *Phys. Fluids A* 1992

Experiments: Finger Width



Glycerin: Red
Silicone Oil: Blue

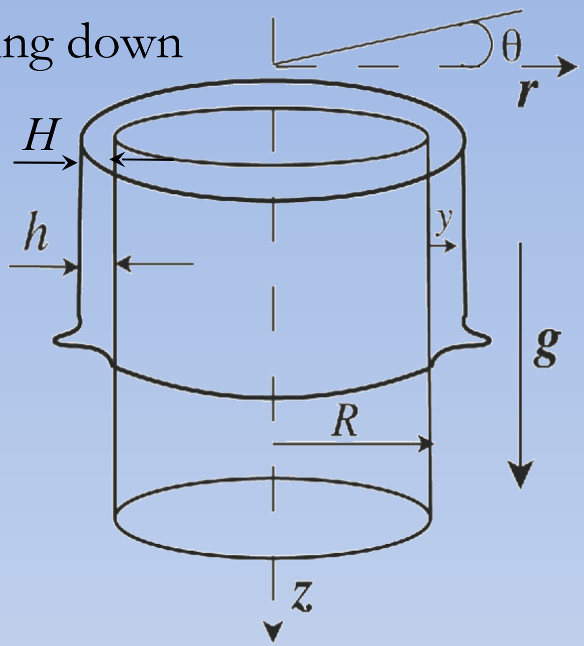
$$\hat{\text{Bo}} = (\rho \; g \, R^2)/\gamma = \text{Bond Number}$$

Finger width is invariant to:

- **Wetting property of the fluids**
- **Cylinder radius**

Derivation of Lubrication Model

Consider the motion of an incompressible thin film moving down the outside of a vertical cylinder.



R : cylinder radius

$h(\theta, z, t)$: fluid free surface

H : upstream film thickness

In the lubrication approximation, assume

$$\varepsilon = H / R << 1,$$

which is **proportional to the substrate curvature**.

$$\mathbf{u} = u\,\mathbf{e}_r + v\,\mathbf{e}_\theta + w\,\mathbf{e}_z$$

Nondimensionalize the free boundary problem using the scalings of Evans, Schwartz, Roy, *Phys. Fluids* 2004:

$$y = \varepsilon R\bar{y}, \quad h = \varepsilon R\bar{h}, \quad z = R\bar{z}, \quad r = R\bar{r}, \quad t = \frac{R}{U}\bar{t},$$

$$u = \varepsilon U\bar{u}, \quad v = U\bar{v}, \quad w = U\bar{w}, \quad p = \rho g H\bar{p}, \quad U = \frac{gH^2}{\nu}.$$

Scaled Free Boundary Problem

Continuity and Navier-Stokes equations:

$$\bar{\Delta} = \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{y}}\left(\bar{r}\frac{\partial}{\partial \bar{y}}\right) + \frac{\varepsilon^2}{\bar{r}^2}\frac{\partial^2}{\partial \theta^2} + \varepsilon^2\frac{\partial^2}{\partial \bar{z}^2}, \quad \bar{\nabla}^2 = \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \bar{z}^2}.$$

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{y}}(\bar{r}\bar{u}) + \frac{1}{\bar{r}}\frac{\partial \bar{v}}{\partial \theta} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0$$

$$\varepsilon^2 \mathrm{Re}\left(\varepsilon\frac{R}{UT}\frac{\partial \bar{u}}{\partial \bar{t}} + \varepsilon\bar{u}\frac{\partial \bar{u}}{\partial \bar{y}} + \varepsilon\frac{\bar{v}}{\bar{r}}\frac{\partial \bar{u}}{\partial \theta} + \varepsilon\bar{w}\frac{\partial \bar{u}}{\partial \bar{z}} - \frac{\bar{v}^2}{\bar{r}}\right) = -\frac{\partial \bar{p}}{\partial \bar{y}} + \varepsilon\bar{\Delta}\bar{u} - \varepsilon^3\frac{\bar{u}}{\bar{r}^2} - \frac{2\varepsilon^2}{\bar{r}^2}\frac{\partial \bar{v}}{\partial \theta}$$

$$\varepsilon^2 \mathrm{Re}\left(\frac{R}{UT}\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\bar{v}}{\bar{r}}\frac{\partial \bar{v}}{\partial \theta} + \bar{w}\frac{\partial \bar{v}}{\partial \bar{z}} + \varepsilon\frac{\bar{u}\bar{v}}{\bar{r}}\right) = -\frac{\varepsilon}{\bar{r}}\frac{\partial \bar{p}}{\partial \theta} + \bar{\Delta}\bar{v} - \varepsilon^2\frac{\bar{v}}{\bar{r}^2} - \frac{2\varepsilon^3}{\bar{r}^2}\frac{\partial \bar{u}}{\partial \theta}$$

$$\varepsilon^2 \mathrm{Re}\left(\frac{R}{UT}\frac{\partial \bar{w}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{w}}{\partial \bar{y}} + \frac{\bar{v}}{\bar{r}}\frac{\partial \bar{w}}{\partial \theta} + \bar{w}\frac{\partial \bar{w}}{\partial \bar{z}}\right) = -\varepsilon\frac{\partial \bar{p}}{\partial \bar{z}} + \bar{\Delta}\bar{w} + 1$$

Boundary conditions:

$$\bar{y} = 0: \quad \bar{u} = \bar{v} = \bar{w} = 0$$

$$\bar{y} = \bar{h}: \quad -\bar{p} + 2\varepsilon\frac{\partial \bar{u}}{\partial \bar{y}} - 2\varepsilon\frac{\partial \bar{h}}{\partial \bar{z}}\frac{\partial \bar{w}}{\partial \bar{y}} - 2\varepsilon\frac{\partial \bar{h}}{\partial \theta}\frac{\partial \bar{v}}{\partial \bar{y}} + O(\varepsilon^2) = -\frac{1}{\varepsilon\,\hat{\mathrm{Bo}}}\left(1 - \varepsilon\bar{h} - \varepsilon\bar{\nabla}^2\bar{h} + O(\varepsilon^2)\right)$$

$$\bar{y} = \bar{h}: \quad \frac{\partial \bar{v}}{\partial y} - \varepsilon\bar{v} + O(\varepsilon^2) = 0$$

$$\bar{y} = \bar{h}: \quad \frac{\partial \bar{w}}{\partial \bar{y}} + O(\varepsilon^2) = 0$$

$$\bar{y} = \bar{h}: \quad \frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\bar{v}}{\bar{r}}\frac{\partial \bar{h}}{\partial \theta} + \bar{w}\frac{\partial \bar{h}}{\partial \bar{z}} = \bar{u}$$

where: $\varepsilon = H/R$, $\mathrm{Re} = RU/\nu$, $\hat{\mathrm{Bo}} = \rho g R^2/\gamma = \varepsilon^{-2}\mathrm{Bo}$ where $\mathrm{Bo} = \rho g H^2/\gamma$.

Lubrication approximation: Assume $\varepsilon << 1$ and $\varepsilon^2 \mathrm{Re} << 1$.

Lubrication Model

Expand the pressure and velocity fields in powers of epsilon:

$$\bar{p} = \varepsilon^{-1} p_0 + p^{(0)} + \varepsilon p^{(1)} + \ldots,$$

$$\bar{\mathbf{u}} = \mathbf{u}^{(0)} + \varepsilon\, \mathbf{u}^{(1)} + \ldots,$$

linearize the free boundary problem and solve for flow variables at $O(1)$ and $O(\varepsilon)$.
Next, substitute flow variables into conservation of mass

$$\left(1 + \varepsilon \bar{h}\right) \frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\partial \bar{Q}_\theta}{\partial \theta} + \frac{\partial \bar{Q}_z}{\partial \bar{z}} = 0,$$

where

$$\bar{Q}_\theta = \int_o^{\bar{h}} \bar{v}\, d\bar{y} = \int_o^{\bar{h}} \left(v^{(0)} + \varepsilon\, v^{(1)}\right) d\bar{y},$$

$$\bar{Q}_z = \int_o^{\bar{h}} \overline{rw}\, d\bar{y} = \int_o^{\bar{h}} \left(1 + \varepsilon\, \bar{y}\right)\left(w^{(0)} + \varepsilon\, w^{(1)}\right) d\bar{y},$$

to obtain an evolution equation for the film height:

$$\left(1 + \underline{\varepsilon\, \bar{h}}\right) \partial_{\bar{t}} \bar{h} + \frac{1}{3} \partial_{\bar{z}} \left(\left(1 + \underline{\varepsilon\, \bar{h}}\right) \bar{h}^3\right) + \frac{\varepsilon}{3\hat{\mathrm{Bo}}} \bar{\nabla} \left(\bar{h}^3\, \bar{\nabla} \left(\underline{1} + \bar{\nabla}^2\right) \bar{h}\right) = 0$$

Gravity **Surface Tension**

$$\bar{\nabla} = \mathbf{e}_\theta \frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial \bar{z}}$$

Experimental Parameters

Cylinder Radius	Slenderness Parameter	Glycerin	Glycerin	Silicone Oil	Silicone Oil
R (cm)	▯	**Re**		**Re**	
0.159	5.3e-1	1.6e-2	0.536	9.9e-3	1.12
0.318	2.7e-1	3.3e-2	2.14	2.0e-2	4.46
0.635	1.3e-1	7.0e-2	8.55	4.1e-2	17.8
0.953	8.9e-2	8.3e-2	19.3	5.7e-1	40.1
1.27	6.7e-2	1.5e-1	34.2	8.42e-2	71.2
Cylinder Radius	Slenderness Parameter	Glycerin	Glycerin	Silicone Oil	Silicone Oil

H : upstream film thickness (= 0.085 ± 0.02 cm)

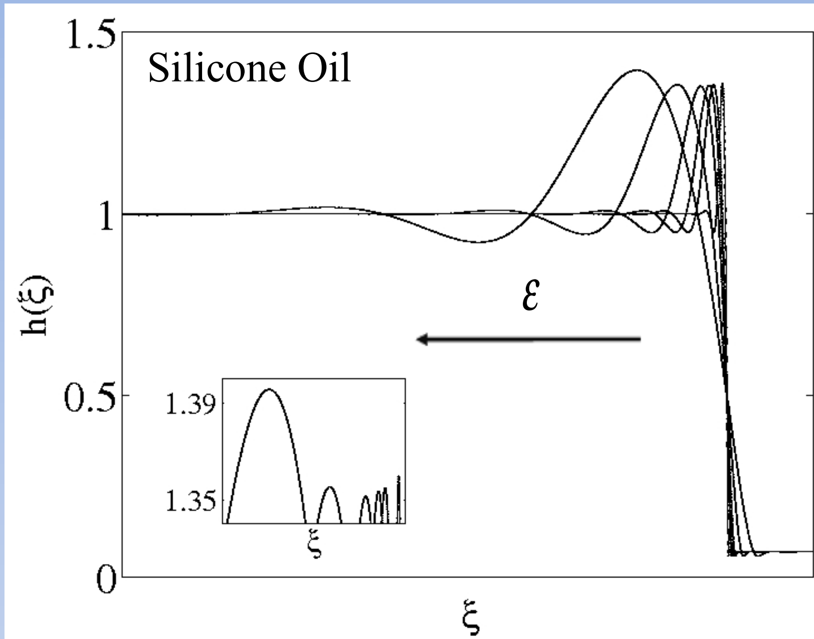
$\varepsilon = H/\mathrm{R} \sim$ substrate curvature

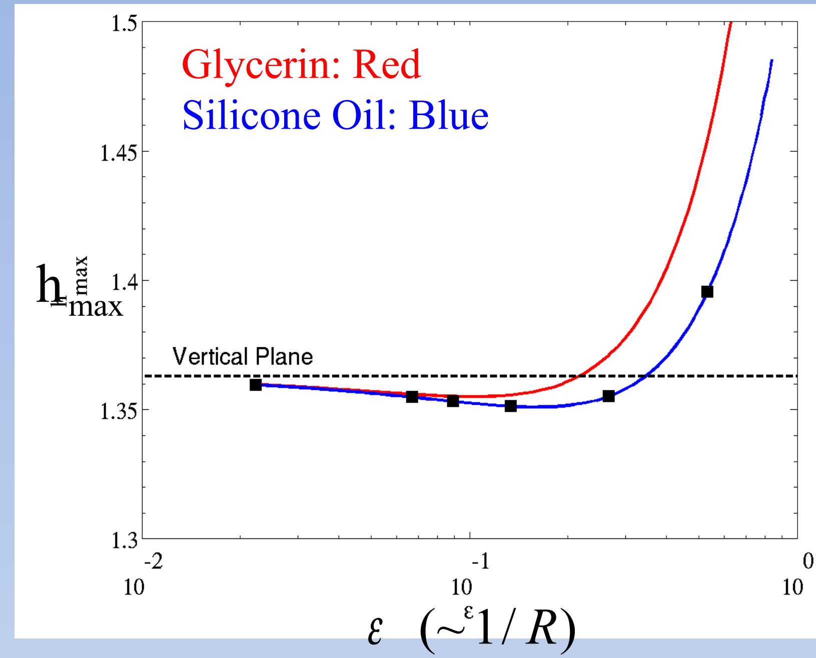
$\mathrm{Re} = \mathrm{R}U/\nu, \quad \mathrm{Bo} = \rho g \mathrm{H}^2/\gamma, \quad \hat{\mathrm{B}}\mathrm{o} = \rho g \mathrm{R}^2/\gamma = \varepsilon^{-2}\mathrm{B}o,$

$\mathrm{Bo}_{\text{glycerin}} = 0.15, \quad \mathrm{Bo}_{\text{silicone oil}} = 0.32.$

- Values of ▯ and Re satisfy conditions to apply lubrication model.
- Wide range of $\hat{\mathrm{B}}\mathrm{o}$ in experiments.

Steady-State Traveling Wave Solutions of Lubrication Model





Details:

- Traveling waves modeled with a precursor film ahead of the contact line, b=0.07.
- Left graph: Parameters taken from experiments.
- Left graph: Characteristic coordinate $\xi =$ z - Ut) is scaled by R.

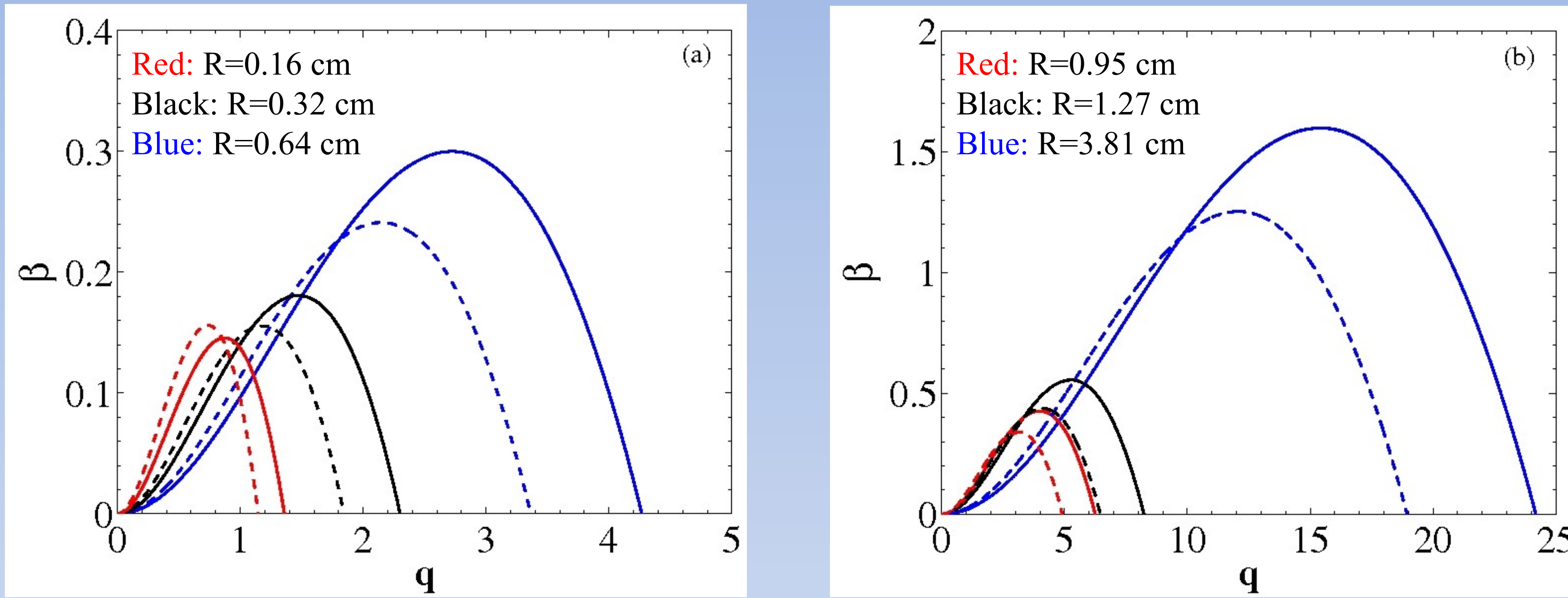
Predictions from model (right graph):

- Influence of substrate curvature negligible for small ε as capillary ridge height and shape converge to that for a traveling wave down a vertical plane.
- When $\varepsilon \geq O(10^{-1})$, substrate curvature and capillary effects influence behavior.

Linear Stability Analysis

Perturbed solution: $h = h_0 + \varphi(\xi)\exp(\beta t + iq\theta)$

Glycerin: Dashed
Silicone Oil: Solid



Experimental parameters used to compute dispersion curves.

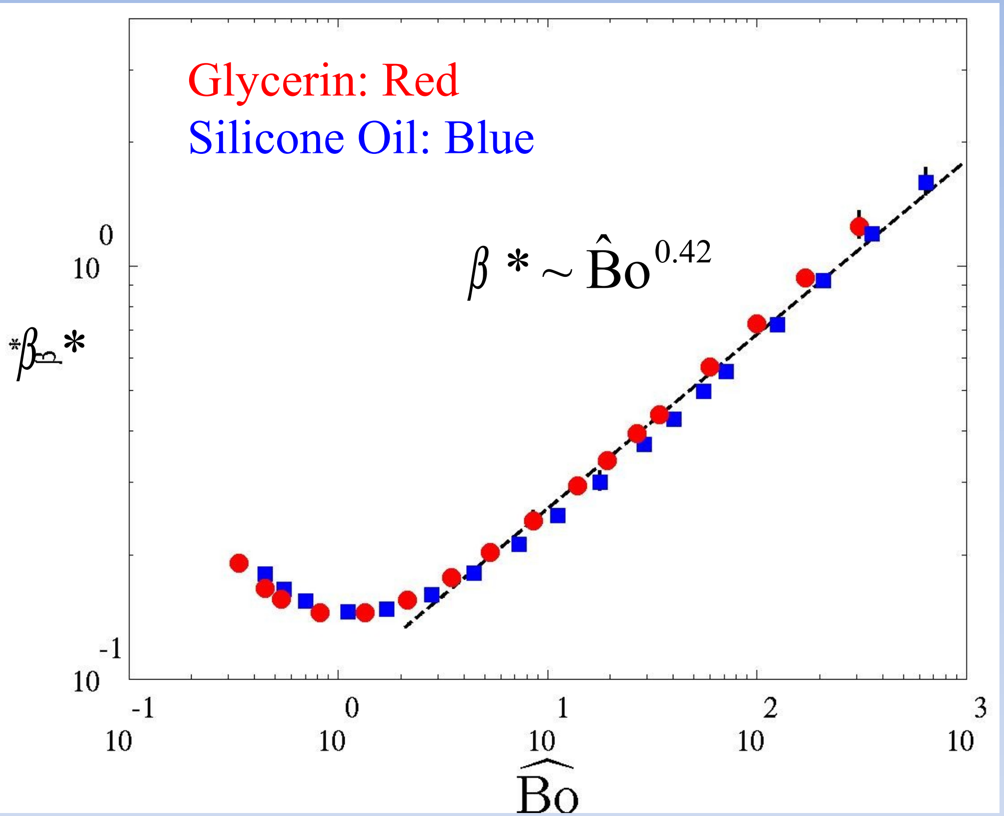
Influence of Capillary Effects

The range of unstable modes, most unstable mode and maximum growth rate are larger for silicone oil.

The contact line for silicone oil is more unstable to fingering than glycerin.

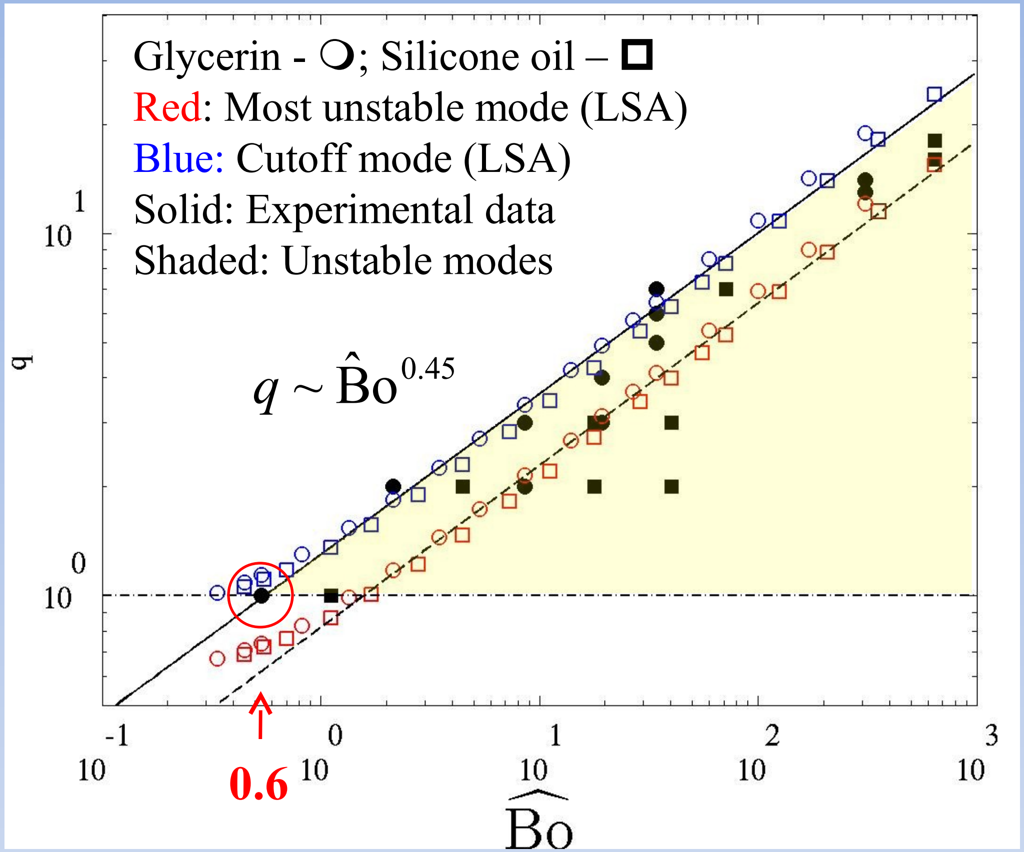
Prediction from Lubrication Model

Maximum Growth Rate



Collapse of data: Maximum growth rate scales with the Bond number for $\hat{Bo} > 1.3$.

Stability: Theory and Experiments



Periodicity of Cylinder Periphery: # of fingers must be integer-valued.

Linear Approximation in : The # of fingers in a pattern and wave number, **q**, are equivalent. For fingers to develop, $q \geq 1$.

Interpretation: The contact line is **stable** when unstable modes are < 1. Otherwise, contact line is **unstable.**

Results: Excellent agreement between linear stability theory and experiments.

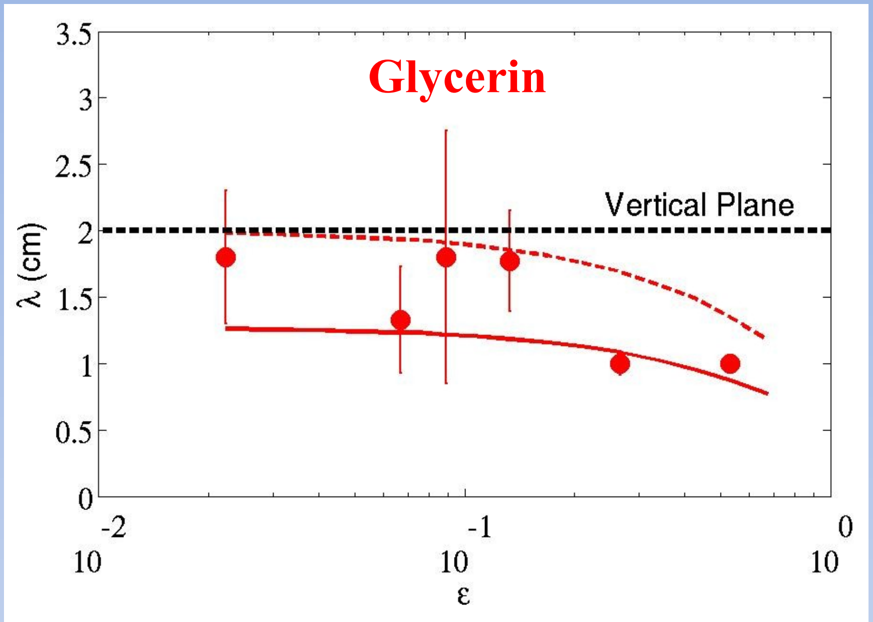
Scaling of q with Bond #:
The # of fingers increases nearly linearly with cylinder circumference.

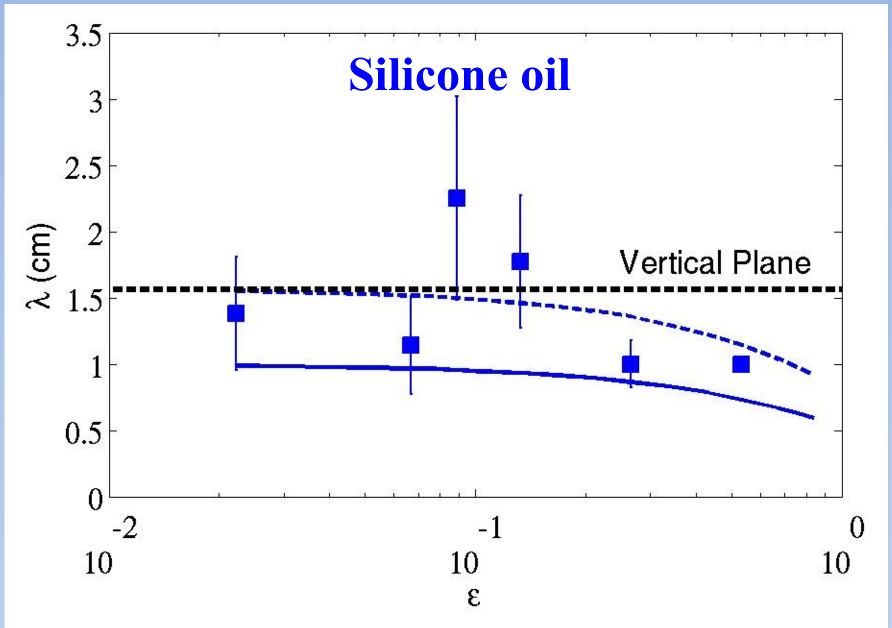
Prediction from Stability Theory: The contact line down a vertical cylinder is **unstable** for

$\hat{\text{Bo}} > 0.6$ or $R > 0.025\sqrt{\gamma / \rho}$

Can we predict when no fingering occurs?

Finger Wavelength: Theory and Experiments $\varepsilon \sim 1/R$





Symbols: Experimental data; **Solid/Dashed colored lines:** Cut-off/Most unstable wavelength from LSA (contact line is stable below solid curve); **Black dashed line:** Most unstable wavelength for a vertical plane, $\lambda = 12.6l, \quad l = (H\gamma / \rho g)^{1/3}$, Spaid and Homsy, *Phys. Fluids* 1996.

Results: **Excellent agreement** between theory and experiments.

Predictions from Model:

- Substrate curvature is negligible when $\varepsilon \leq O(10^{-2})$, and influence behavior when $\varepsilon = O(10^{-1})$.

- Larger surface tension increases finger wavelength (□glycerin > □ silicone oil)

Conclusions

We've presented an experimental and analytical study on the dynamics of an axisymmetric contact line moving down a vertical circular cylinder.

Model accurately predicts the wavelength and number of fingers that form in experiments for a variety of cylinder sizes and for two fluids.

Model Predictions:

- Substrate curvature is a negligible effect when $\varepsilon \leq O(10^{-2})$ (shape and stability of the contact line down a vertical cylinder matches that down a vertical plane) and substrate curvature is an important effect when $\varepsilon = O(10^{-1})$.

- Model identifies a critical Bond number for fingering.

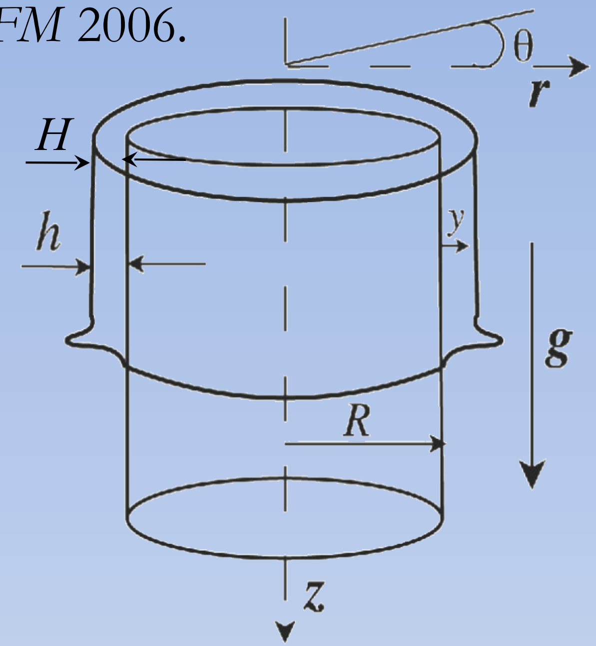
Future Directions:

- Analyze fingering for thicker films when H ~ R (flow down a wire). Non-trivial!
- Study effect of inertia on fingering dynamics.

Is there a model that allows H~R?

Craster & Matar, *On viscous beads flowing down a vertical fibre, JFM* 2006.
Derive a 1D model in the long-wavelength limit with

$$\varepsilon = (R + H)/L << 1,$$ ⟵ promising



L is the capillary length

$$L = \gamma/(\rho g(R + H))$$

or equivalently the Bond number << 1.

Naïve Idea: Develop a 2D model using C&M's scalings.

$$r = (\mathrm{R} + \mathrm{H})\bar{\mathrm{r}}, \quad \mathrm{z} = L\bar{\mathrm{z}}, \quad \mathrm{t} = \frac{U\bar{\mathrm{t}}}{L},$$

Free surface at $\boldsymbol{r = S = R + h}$

$$u = \varepsilon U\bar{\mathrm{u}}, \quad \bar{\mathrm{v}} = ?, \quad \mathrm{w} = U\bar{\mathrm{w}}, \quad \mathrm{p} = \rho g L\bar{\mathrm{p}}, \quad U = \frac{\mathrm{g(R + H)}^2}{\nu}.$$

$$\mathbf{u} = u\,\mathbf{e}_r + v\,\mathbf{e}_\theta + w\,\mathbf{e}_z$$

The Continuity Equation sets the scaling on the azimuthal velocity.

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \qquad \frac{\varepsilon U}{R + H}\left(\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}}{\bar{r}}\right) + \frac{?}{R + H}\frac{1}{\bar{r}}\frac{\partial \bar{v}}{\partial \theta} + \frac{U}{L}\frac{\partial \bar{w}}{\partial \bar{z}} = 0$$

For all the terms to be O(1) requires that $v = \varepsilon U\bar{v}$!

Leading Order Free Boundary Problem

Continuity and Navier-Stokes equations (dropping hat notation):

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0,$$

$$\frac{\partial p}{\partial r} = 0,$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = 0,$$

$$\frac{\partial \bar{p}}{\partial \bar{z}} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + 1.$$

with boundary conditions at the free surface $r = S$:

$$p = \frac{1}{\sqrt{S^2 + (S_\theta)^2}} + \frac{(S_\theta)^2 - S_{\theta\theta} S}{\left(\sqrt{S^2 + (S_\theta)^2}\right)^3} - \varepsilon^2 \mathrm{S}_{\mathrm{zz}},$$

$$\frac{\partial w}{\partial r} - \frac{S_\theta}{S^2} + \frac{\partial w}{\partial \theta} = 0,$$

$$\frac{S_\theta}{S}\left(2\frac{\partial u}{\partial r} - \frac{2}{S}\frac{\partial v}{\partial \theta} + \frac{1}{S}u\right) - \frac{S_z}{S}\left(S_\theta\frac{\partial w}{\partial r} + \frac{\partial w}{\partial \theta}\right) + \left(\frac{1}{S}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{S}\right)\left(1 - \left(\frac{S_\theta}{S}\right)^2\right) = 0,$$

$$S_t + \mathrm{w}\mathrm{S}_{\mathrm{z}} + \frac{v}{S}S_\theta = u$$

Leading Order Free Boundary Problem

Continuity and Navier-Stokes equations:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0,$$

$$\frac{\partial p}{\partial r} = 0,$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = 0,$$

$$\left.\right\} \quad p = p(z,t)$$

$$\frac{\partial \bar{p}}{\partial \bar{z}} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + 1.$$

Normal BC at $r = S$:

$$p = \frac{1}{\sqrt{S^2 + (S_\theta)^2}} + \frac{(S_\theta)^2 - S_{\theta\theta} S}{\left(\sqrt{S^2 + (S_\theta)^2}\right)^3} - \varepsilon^2 S_{zz},$$

Depends only on z, t.

For fingering, need azimuthal dependence in free surface, S. Mathematically inconsistent!

Naïve!

Thank you!

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Thomas Witleski

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