Counting Rational Points on Certain Pfaffian Sets

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Let $X \subseteq \mathbb{R}^n$ be definable in an o-minimal expansion of $\overline{\mathbb{R}}$ and consider $|X \cap \mathbb{Q}^n|$.

Guiding Principle:
If X contains "too many" rational points, then it must contain an infinite connected semialgebraic set.

Turn this around:
Consider $X^{\text{trans}} := X \setminus X^{\mathrm{alg}}$, the transcendental part of X, where X^{alg} is the union of all infinite, connected, semialgebraic subsets of X.
We investigate when X^{trans} does not contain "too many" rational points.
But it is not a finitary/infinitary question - consider $\left|\mathrm{graph}(2^x) \cap \mathbb{Q}^2\right|$. Not finite but 2^x is a transcendental function.

Instead categorise rational points by *height*: $H(\frac{a}{b}) := \max\{|a|, |b|\}$.
So, for a given height $T \in \mathbb{N}$, attention is restricted to at most T^2 points, $0, 1, \ldots, T, \ldots, \frac{1}{T}, \ldots, \frac{T}{T}$.
$\mathbb{Q}^n(T) := \{\overline{q} \in \mathbb{Q}^n | H(q_i) \leq T\}; \qquad |\mathbb{Q}^n(T)| \leq T^{2n}$.
We then count $|X^{\text{trans}} \cap \mathbb{Q}^n(T)|$ and see how fast it grows with T.

Theorem (Pila-Wilkie 2006)

Let $X \subseteq \mathbb{R}^n$ be definable in an o-minimal expansion of $\overline{\mathbb{R}}$.
For all $\varepsilon > 0$, there exists $c(X, \varepsilon) > 0$ such that, for all (sufficiently large) $T \in \mathbb{N}$,

$$\left|X^{trans} \cap \mathbb{Q}^n(T)\right| \leq cT^{\varepsilon}.$$

Best possible statement for o-minimal expansions of $\overline{\mathbb{R}}$ in general (counterexample curve in $\mathbb{R}_{\text{an}}$).

However, proposed improvement for $\mathbb{R}_{\exp}$:

Wilkie's Conjecture (2006)

For all sets X definable in $\mathbb{R}_{\exp}$, there exist $c(X), \gamma(X) > 0$ such that

$$\left| X^{\text{trans}} \cap \mathbb{Q}^n(T) \right| \leq c(\log T)^{\gamma}, \text{ for } T \geq e.$$

Remark: All results in this talk also apply to bounds on the density of algebraic points in a fixed number field (using the absolute multiplicative height).

Theorem (Pila 2007)

For any $X = graph(f)$, where $f : I \longrightarrow \mathbb{R}$ is a transcendental Pfaffian function on an interval $I \subseteq \mathbb{R}$, there exist $c(X), \gamma(X) > 0$ such that $\left|X \cap \mathbb{Q}^2(T)\right| \leq c(\log T)^{\gamma}$.

2 ingredients in the proof.

Proposition (Pila 2007)

Let $f : I \rightarrow \mathbb{R}$ be D-times differentiable on an interval I with $\frac{1}{T^2} < l(I) < 2T$, where $D := \frac{(d+1)(d+2)}{2}$, for some $d \geq 1$ and $T \geq e$. Suppose $|f'| \leq 1$ and, for each order $j = 1, \ldots, D$, $f^{(j)}$ is either identically 0 or non-vanishing. Then $X \cap \mathbb{Q}^2(T)$ is contained in at most $cD(logT)T^{\frac{32}{3(d+3)}}$ real algebraic curves of degree $\leq d$.

The second ingredient:

For a Pfaffian function f, we have effective Khovanskii bounds on the number of connected components of:
$V(f)$, the zero set of f, and on
$X \cap V(P)$, the graph of f intersected with the zero set of a polynomial P,
which depend
on the complexity of f - the number of variables of f, the number of differential equations r in the system describing f, the degrees of the polynomials in that system
and
(polynomially) on the degree d of P.

Idea of Proof of Pila's Theorem

Theorem (Pila 2007)

For any $X = \text{graph}(f)$, where $f : I \longrightarrow \mathbb{R}$ is a transcendental Pfaffian function on an interval $I \subseteq \mathbb{R}$, there exist $c(X), \gamma(X) > 0$ such that $\left|X \cap \mathbb{Q}^2(T)\right| \leq c(\log T)^{\gamma}$.

Idea of Proof.

See board... $\square$

Theorem (Jones-T. 2010)

For any $X = graph(f)$, *where* $f\colon I \longrightarrow \mathbb{R}$ *is a transcendental function implicitly defined from Pfaffian functions on an interval* $I \subseteq \mathbb{R}$, *there exist* $c(X), \gamma(X) > 0$ *such that*

$$|X \cap \mathbb{Q}^n(T)| \leq c(\log T)^{\gamma}.$$

Proof.

First must establish analogous bounds on the number of connected components of $V(f), V(f'), \ldots, V(g), V(g'), \ldots$ for f defined implicitly from Pfaffian functions (and g its inverse).
Then the proof follows the method of Pila's theorem. $\square$

Corollary (Jones-T. 2010)

For any $X = graph(f)$, where $f : I \longrightarrow \mathbb{R}$ is existentially definable in $\mathbb{R}_{Pfaff}$, there exist $c(X), \gamma(X) > 0$ s.t. $|X^{trans} \cap \mathbb{Q}^n(T)| \leq c(\log T)^{\gamma}$.

Proof.

By methods of Wilkie, for such a function f there are intervals $I_1, \ldots, I_m$, covering I up to a finite set, such that, on each I_i, f is implicitly defined. Apply the above to each I_i in turn. □

Moreover, this bound will hold for any function definable in any model complete reduct of $\mathbb{R}_{\text{Pfaff}}$ - in particular in $\mathbb{R}_{\exp}$.

Corollary (Jones-T. 2010; also shown by Butler)

Wilkie's Conjecture holds for any 1-dimensional set X.

Idea:
Contain the rational points on a surface X in a controlled number of algebraic surfaces. Then count points on intersections.

Theorem (Pila 2009)

Suppose $X \subseteq (0,1)^3$ has dimension 2 and admits a mild parameterization; then $X \cap \mathbb{Q}^3(T)$ is contained in at most $K_1(\log T)^{K_2}$ algebraic surfaces of degree $\lfloor(\log T)^2\rfloor$, for some $K_1(X), K_2(X) > 0$.*

*Mild parameterization:
A surface X has a mild parameterization if there exist finitely many C^∞ functions $\phi_1,\ldots,\phi_l\colon (0,1)^2 \to X$ such that $X = \bigcup_{i=1}^{l} \phi_i((0,1)^2)$ and the nth $(= (n_1+n_2)$th) order derivatives of each ϕ_i are bounded by $n_1!n_2!(Bn^C)^n$, for some $B, C > 0$.

We study the intersection of definable surfaces with algebraic surfaces.
Consider X, the graph of a function $f : U \longrightarrow \mathbb{R}$ implicitly defined from Pfaffian functions on an analytic cell $U \subseteq \mathbb{R}^2$.
We would like to bound the density of rational points on sets of the form $X \cap V(P)$, for $P \in \mathbb{R}[X_1, X_2, X_3]$.

Proposition (Jones-T. 2010)

For such a surface X, there exist $c(X), \gamma(X) > 0$ and a polynomial $Q \in \mathbb{R}[X]$ of degree $N(X)$ such that, for any polynomial $P : \mathbb{R}^3 \to \mathbb{R}$ of degree d, $\left|(X \cap V(P))^{trans} \cap \mathbb{Q}^3(T)\right| \leq cQ(d)(\log T)^{\gamma}$.

Proof of Proposition

Proposition (Jones-T. 2010)

For such a surface X, there exist $c(X), \gamma(X) > 0$ and a polynomial $Q \in \mathbb{R}[X]$ of degree $N(X)$ such that, for any polynomial $P : \mathbb{R}^3 \to \mathbb{R}$ of degree d, $\left|(X \cap V(P))^{trans} \cap \mathbb{Q}^3(T)\right| \leq cQ(d)(\log T)^{\gamma}$.

Idea of Proof.

See board... $\square$

Some consequences of combining this result with Pila's Theorem:

1st Corollary (Jones-T. 2010)

If $X \subseteq \mathbb{R}^n$ is a surface definable in $\mathbb{R}_{resPfaff}$, the real field expanded by all restricted Pfaffian functions, then there exist $c(X), \gamma(X) > 0$ such that $|X^{trans} \cap \mathbb{Q}^n(T)| \leq c(\log T)^{\gamma}$.

Proof.

All sets definable in $\mathbb{R}_{\text{resPfaff}}$ are subanalytic and therefore admit mild parameterization. (Even a definable mild parameterization - Jones-Miller-T. 2009.) $\square$

2nd Corollary (Jones-T. 2010)

Wilkie's Conjecture holds for any surface X which admits a mild parameterization.

Let $\alpha, \beta, 1$ be linearly independent over $\mathbb{Q}$. Consider

$$X := \{(x, y, z) \in (0, \infty)^3 | z = x^\alpha y^\beta\}.$$

X^{alg} is empty. We can find a mild parameterization that gives us, for some $c(X), \gamma(X)$, and $T \geq e$, $\left|X \cap \mathbb{Q}^3(T)\right| \leq c(\log T)^\gamma$.

Proposition

Suppose $(x_1, \ldots, x_k), (y_1, \ldots, y_k)$ are two multiplicatively independent k-tuples of rationals, where $k > \gamma$. At least one of $x_i^\alpha y_i^\beta$ is irrational.

Irrationality Result ctd.

$$X := \{(x,y,z) \in (0,\infty)^3 | z = x^\alpha y^\beta\}. \ \left|X \cap \mathbb{Q}^3(T)\right| \leq c(\log T)^\gamma.$$

Proposition

Suppose $(x_1,\ldots,x_k),(y_1,\ldots,y_k)$ are two multiplicatively independent k-tuples of rationals, where $k > \gamma$. At least one of $x_i^\alpha y_i^\beta$ is irrational.

Proof.

Suppose $x_i^\alpha y_i^\beta$ are all rational. Then all $(x_i, y_i, x_i^\alpha y_i^\beta) \in X \cap \mathbb{Q}^3$.
Moreover, for any k-tuples $(m_1,\ldots,m_k),(n_1,\ldots,n_k) \in \mathbb{N}^k$, we also have $(x_i^{m_i}, y_i^{n_i}, x_i^{\alpha m_i} y_i^{\beta n_i}) \in X \cap \mathbb{Q}^3$.
By considering $k+1$ pairs of tuples $\bar{m},\bar{n}$, we have that $\left|X \cap \mathbb{Q}^3(T)\right| \geq c(\log T)^{k+1}$, which is a contradiction. $\square$