Financial Crises and Contagion: Dynamical Systems Approach

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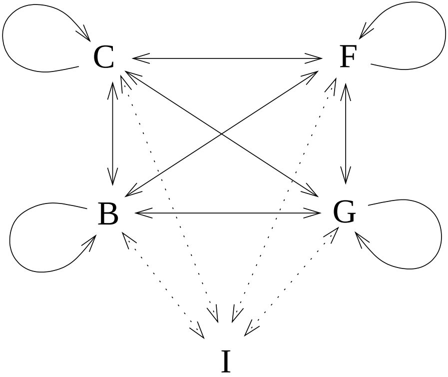
Outline

Goal: dynamic modeling of financial crises and systemic risk

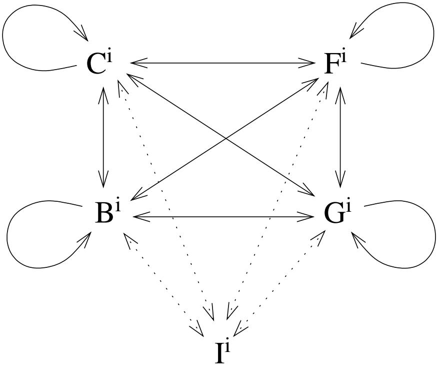
1. Single Economy: w/ **R. Douady**
   - Cause: breakage of stability $\Longrightarrow$ bifurcation
   - Effect: contagion, systemic risk $\Longrightarrow$ recurrence, chaos
   - Predicting a crisis: Market Instability Indicator
   - Suggested remedies
2. Multiple Economies: w/ **G. Castellacci**
   - Contagion from one economy to another
   - Quantitative definition of contagion
   - Suggested remedies

Single Economy Five Agent Model



C	Consumers
F	Firms
B	Banks
I	Investors
G	Government

Generalized Single Economy Five Agent Model



Agents of Economy i

C^i	Consumers
F^i	Firms
B^i	Banks
G^i	Government
I^i	Investors restricted to Economy i

Figure : Combined flow of funds among five agents in economy i

Flows of Funds: Scheduled vs. At-will

- **Scheduled Cash Flows**:
  - Coupons
  - Installments, minimum credit card payments
  - Salaries
  - Contributions to pension plans
  - Taxes
- **At-will Cash Flows**: variable
  - Equity investments
  - Debt investments (loans, bonds)
  - Dividends
  - Consumption

Both are *variable* and subject to *dynamic* relations

More Flows of Funds: Contingent & International

- **Contingent Cash Flows**:
  - Quantitative Easing
  - Derivative Payoffs, e.g. CDS payouts
- **International Debt Investment**:
  - Interbank lending and investment
  - Investment in sovereign debt
  - Central banks' lending to foreign banks
- **International Consumption and Trade**:
  - Direct consumption of foreign goods and services
  - International trade between firms

Flow of Funds for Two Economies

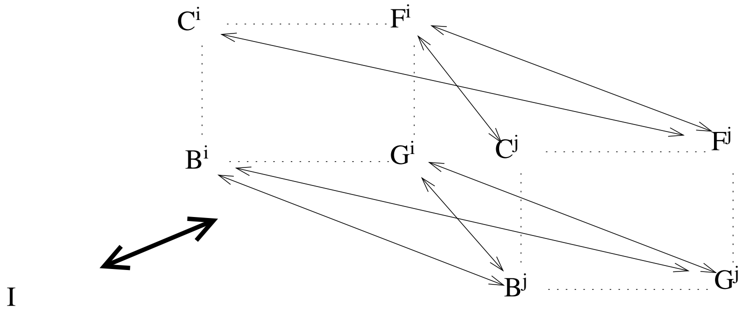


Figure : Flow of funds between economies *i* and *j*

Stage 1 Contagion

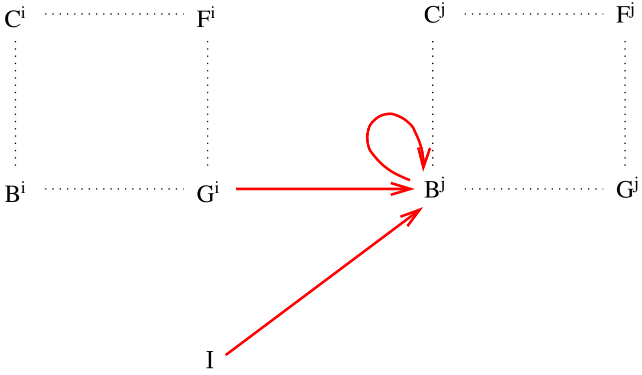


Figure : Contagion from debtor *i* to creditor *j* inside eurozone.

- Contagion of "reduced flow of funds"

Stage 2 Contagion

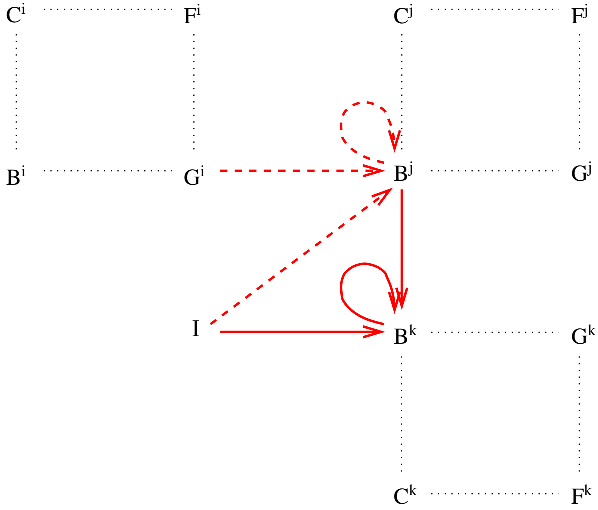


Figure : Contagion spills out of the eurozone

Early Bailout

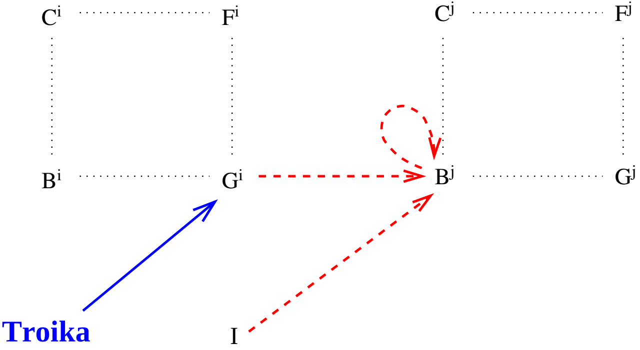


Figure : Earlier stage of the eurozone crisis

Wealth Decomposition

$w_i(t)$ = Wealth of Agent i at time t, ($i = 1, \cdots, 5$ for C, F, B, G, I)

- Equity / Debt split
  - $w_i(t) = E_i(t) + D_i(t)$
  - $E_i(t)$ = Equity value
  - $D_i(t)$ = Debt value
- Liquid Asset / Invested Asset split
  - $w_i(t) = L_i(t) + K_i(t)$
  - $L_i(t)$ = Liquidities: cash, cashables $\Longrightarrow$ produces no income
  - $K_i(t)$ = Invested Assets: financial securities, property, equipment

    $\Longrightarrow$ produces capital gain

Wealth Dynamics

- Debt: $D_i(t+1) = (1 + r_i(t))D_i(t) + \tilde{\Delta} D_i(t+1)$
  - $r_i(t)$ = average interest rate on debt of i at t
  - $\tilde{\Delta} D_i(t)$ = new loans - capital reimbursement
- Invested Asset: $K_i(t+1) = (1 + \gamma_i(t))K_i(t) + \tilde{\Delta} K_i(t+1)$
  - $\gamma_i(t)$ = internal growth factor (IRR)
  - $\tilde{\Delta} K_i(t)$ = new investment - realization
- Liquidities: $L_i(t+1) = L_i(t) + \sum_{j \neq i}^{n} F_{ij}(t) - \sum_{k \neq i}^{n} F_{ki}(t) - \tilde{\Delta} K_i(t)$
  - $F_{ij}(t)$ = fund transferred from j to i at t
  - Can be seen as an "investment" with returns $F_{ji}(s)$, $s > t$
  - $F_{ii}(t) := \gamma_i(t) K_i(t)$
- $w_i(t+1) = w_i(t) + \sum_{j=1}^{n} F_{ij}(t) - \sum_{k \neq i}^{n} F_{ki}(t)$

Wealth Constraints

- Positive liquidities
  - $L_i(t) \geq 0$
  - Negative liquidities $\Longrightarrow$ *debt increase*
- Maximum convertibility rate
  - $|\tilde{\Delta} K_i(t+1)| \leq \kappa_i(t) K_i(t)$
  - There is a limit to converting invested assets to/from liquidities
- Borrowing capacity constraint
  - $D_i(t) \leq D_{i\,\max}(t)$: one cannot borrow forever
  - $D_{i\,\max}(t)$ depends on $w_i(t)$ and on market conditions
  - $(1 + r_i(t))D_i(t) > D_{i\,\max}(t+1) \Longrightarrow$ default, bankruptcy

Assumptions on Variables

- Each $F_{ji}(t)$ produces $F_{ij}(s)$ $(s > t)$ with uncertainty
- Under normal (= non-crisis) times,
  - $r_i(t)$, $\gamma_i(t)$ are continuous
  - $\tilde{\Delta} K_i(t)$, $\tilde{\Delta} D_i(t)$, $\Delta L_i(t)$ are continuous
  - $L_i(t)$, $K_i(t)$, $D_i(t)$ are processes with continuous sample paths
- During a crisis, above not necessarily hold
  - Violent changes in variables can lead to a crisis

Maximizing Benefit I

- $U(x)$ is a utility function on gain x
  - $U : [a, b] \longrightarrow \mathbb{R}, \quad a < 0 < b$

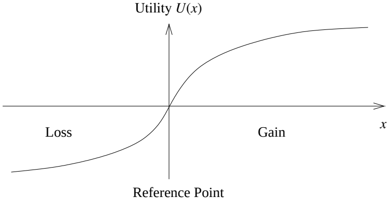


Figure : Convex for losses, concave for gains

- $\mathbb{P}$: probability measure, $F(x) := \mathbb{P}[X \leq x]$
- Expected Utility Theory
  - $E[U(X)] = \int_{\mathbb{R}} U(x)\, dF(x)$

Maximizing Benefit II

- Cumulative Prospect Theory: Subjective Utility (SU)
  - Weighting function: $W = \mathbb{1}_{[a,0)}\, W^{-} + \mathbb{1}_{(0,b]}\, W^{+}$
    $\Longrightarrow$ measures attitude toward risk

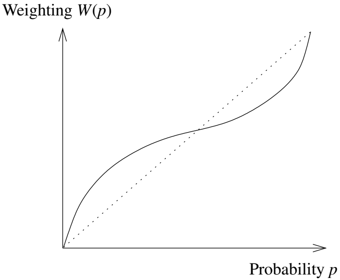


Figure : Overreact to unlikely event, magnifying fear factor

  - $\mathrm{SU}[X] = \int_{\mathbb{R}} U(x)\, W'(F(x))\, dF(x)$

Non Linear Programming Problem

- Apply this to each i for each $[t, t+1]$
  - $U_i(x), \quad \mathbb{P} = \mathbb{P}_t$ w/ $F_t(x) = \mathbb{P}_t[X_i \le x]$
  - $\mathrm{SU}_{i,t}[X]$ = Subjective Utility of $U_i(x)$ at t
  - Net Subjective Utility (Investment) := SU (NPV of Investment)

$$\mathrm{NSU}_{i,t}\left(F_{ji}(t)\right) = \mathrm{SU}_{i,t}\left[\sum_{t<s_l\le T} D(t, s_l)F_{ij}(s_l) - F_{ji}(t)\right]$$

- NLP: Max $z_i = \sum_{j=1}^{n} \mathrm{NSU}_{i,t}\left(F_{ji}(t)\right)$ sub. to
  - $L_i(t) \ge 0$
  - $|\tilde{\Delta}K_i(t+1)| \le \kappa_i(t)K_i(t)$
  - $\tilde{\Delta}D_i(t+1) \le D_{i\,\max}(t+1) - (1 + r_i(t))D_i(t)$
  - $1 \le i \le n, \; t \ge 0$

Optimal Investment: Equilibrium State

- NLP with n objective functions, $3n$ constraints
- $F^*_{ij}(t)$ = the optimal solution, $1 \le i, j \le n$
- Obtain *Random* dynamical system $f(X^*(t)) := X^*(t+1)$ where $X_i(t) = (L_i(t), K_i(t), D_i(t)) \in \mathbb{R}^3, \quad X = (X_1, X_2, \ldots, X_n)$
- Constraints produce *nonlinear dynamics*
  - In a crisis, constraints tend to be *saturated*
    $\Rightarrow$ the dynamics doesn't depend on U_i
  - High leverage makes debt ↑, borrowing capacity ↓
    $\Rightarrow$ hit the constraints
  - Myopic risk estimation
    $\Rightarrow$ short-term statistics extrapolated to long-term risk

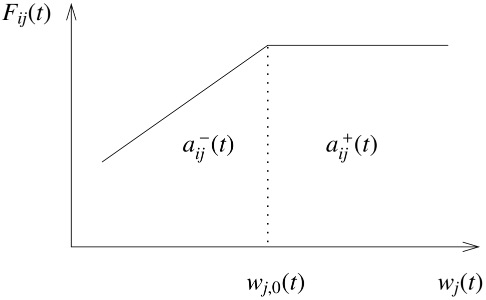
Perturbation Analysis

- From random to deterministic
  - Take non-random part $\bar{f}$ of f and rescale X^* to constant dollar X
  - We get *deterministic* dynamical system $X(t+1) = \bar{f}(X(t))$
  - If $\bar{f}$ becomes unstable, so does f
- There is an equilibrium (fixed point)$\tilde{X} = (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$
  - Diminishing marginal utility in closed economy
  - Every agent has become as rich as it can be
  - Brouwer fixed point theorem on a compact convex set
- Stable equilibrium (attracting fixed point): $\bar{f}(\tilde{X}) = \tilde{X}$
  - Stable wealth stabilizes NLP constraints
  - Small changes in constraints preserve optimal solution

Elasticity Coefficient I

- Drop "overline" from $\bar{f}$: $X(t+1) = f(X(t))$
- df is $3n \times 3n$: $f(X(t) + \delta X) \approx f(X) + df(X(t))\delta X$
- $\delta X_i = (\delta L_i, \delta K_i, \delta D_i)$, $\delta w_i = \delta L_i + \delta K_i$
- Derive a "reduced" Jacobian B:
  - $\delta X'(t+1) = df(X(t))\delta X(t) = (\delta L_i'(t+1), \delta K_i'(t+1), \delta D_i'(t+1))$
  - $\delta L_i'(t+1) + \delta K_i'(t+1) = \delta w'(t+1) \equiv B(X(t))\delta w(t)$
  - $\delta w'(t+1) = B(X(t))\delta w(t)$
- Define *Elasticity Coefficient*: $a_{ij} = a_{ij}^{+}(t)$ or $a_{ij}^{-}(t)$
  - $a_{ij}^{+}(t) = \lim_{\Delta w_j \to 0^+} \dfrac{F_{ij}(w_j(t) + \Delta w_j) - F_{ij}(w_j(t))}{\Delta w_j}$
  - $a_{ij}^{-}(t) = \lim_{\Delta w_j \to 0^-} \dfrac{F_{ij}(w_j(t) + \Delta w_j) - F_{ij}(w_j(t))}{\Delta w_j}$

Elasticity Coefficient II



- Different sign of $\Delta w_j(t)$ yields different reaction of $F_{ij}(t)$:
  - Pre-Crisis C: failure to pay vs. no extra payment/savings
  - Post-Crisis B: credit reduction vs. hoarding cash
  - Post-Crisis F: layoff vs. hire freeze

Market Instability Indicator

- Elasticities vs. reduced Jacobian $B(X(t))$:
  - $b_{ii} = 1 + a_{ii} - \sum_{k \neq i}^{n} a_{ki}$
  - $b_{ij} = a_{ij}$ for $i \neq j$
  - High leverage implies high elasticities
- *Market Instability Indicator*

$$I(t) = \text{Max Eigenvalue of } B(X(t)) = \rho\,(B(X(t)))$$

  - This is *not* a Lyapunov exponent
  - $I(t) < 1$: perturbations of the system tend to be absorbed
  - $I(t) > 1$: small perturbations tend to increase when propagating $\Longrightarrow$ Domino effect: possible Financial Crisis
- $I(t)$ can be empirically observed
  - Lagged correlations of historical series of flow of funds

Financial Crisis: Breakage of Stability I

NLP with reduced borrowing capacity:

- Maximize $z_i = \sum_{j=1}^{n} \mathrm{NSU}_{i,t}\left(F_{ji}(t)\right)$ subject to
  - $L_i(t) \geq 0$
  - $|\tilde{\Delta} K_i(t+1)| \leq \kappa_i(t) K_i(t)$
  - $\tilde{\Delta} D_1(t+1) \leq D_{1\,\max}(t+1) - \mu - (1 + r_1(t))D_1(t)$
  - $\tilde{\Delta} D_i(t+1) \leq D_{i\,\max}(t+1) - (1 + r_i(t))D_i(t)$
  - $2 \leq i \leq n,\ t \geq 0$

  $\Longrightarrow$ obtain $\{f_\mu\}$

- Perturb f by $\{f_\mu\}$ to get new equilibrium $\{\tilde{X}_\mu\}$
  - As leverage increases so do entries of B_μ ($\Longleftarrow$ elasticities)
  - Hence eigenvalues of B_μ increase
  - Even a small default at $\tilde{X}_\mu$ will break the stability: $I(t) > 1$

Financial Crisis: Breakage of Stability II

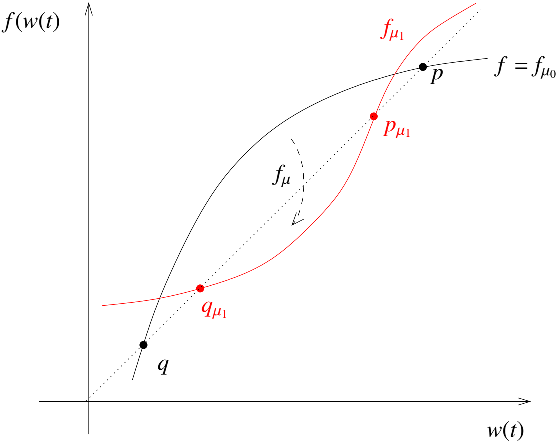
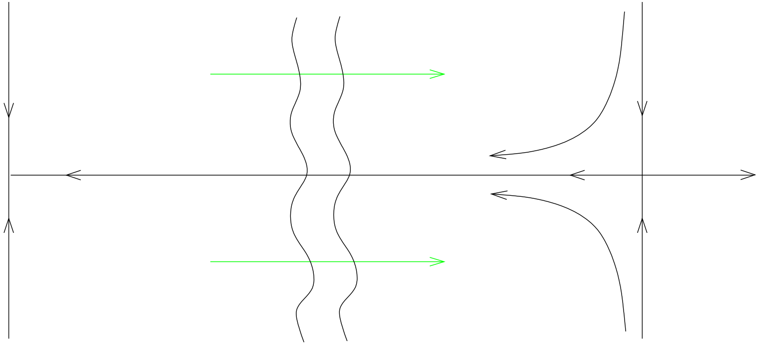


Figure : One dimensional illustration of stability change

Evolution of 2007-2009+ Crisis I

- Cause: breakage of stability $\Longrightarrow$ bifurcation
- Effect: contagion, systemic risk $\Longrightarrow$ recurrence, chaos
  - Securitization interconnected agents
  - "Default" spread along the feedback loop
  - Chaos in the financial crisis
- Remedy: getting out of recession $\Longrightarrow$ QE etc.
  - Default set in: bailouts, loan restructuring, pay cut etc.
  - Agents minimize spending: new $f(\tilde{Y}) = \tilde{Y}$
  - $\tilde{Y}$ is a recession $\Longrightarrow$ Eigenvalues of $B(\tilde{Y}) < 1$
  - Break the equilibrium by raising elasticities: QE etc.
  - Targeted fund allocation is necessary: no random handing out

Evolution of 2007-2009+ Crisis II



Global Recession
Deflation

Gov. action to
prevent deflation
(quant. easing)

Broken
Market Equilibrium

⟶ recurrence

- Government takes action to stay away from deflation (sink)

Agents of Global Economy I

G is a global economy consists of s subeconomies

- Economy k has n_k agents: G has $n = \sum_{k=1}^{s} n_k$ agents
- $w(t) = (w_1(t), w_2(t), \ldots, w_n(t))$: the *global wealth vector*
- For subeconomy k,
  - $w^k(t) = \left(w_1^k(t), w_2^k(t), \ldots, w_{n_k}^k(t)\right)$: the wealth
  - $w_j^k(t)$ is the wealth of agent j at t
  - $w_i(t) = w_j^k(t)$ if $i = N(k) + j$, $N(k) = \sum_{l=1}^{k-1} n_l$
  - $F_{N(k)+i,N(k)+j}(t) = F_{ij}^k(t)$
  - $b_{N(k)+i,N(k)+j}(t) = b_{ij}^k(t)$, $B^{(k)}(t) = \left(b_{ij}^k(t)\right)$ is the Jacobian matrix
  - $a_{N(k)+i,N(k)+j}(t) = a_{ij}^k(t)$, $A^{(k)}(t) = \left(a_{ij}^k(t)\right)$ is the elasticity matrix

Agents of Global Economy II

Between two economies k and l,

- $F^{kl}_{ij}(t) = F_{N(k)+i,N(l)+j}(t)$
  - Flow of funds from agent j of economy l to agent i of economy k at time t
- $a^{kl}_{ij}(t) = a^{kl+}_{ij}(t)$ or $a^{kl-}_{ij}(t)$,
  - $a^{kl+}_{ij}(t) = \lim_{\Delta w^l_j \to 0^+} \dfrac{F^{kl}_{ij}(w^l_j(t) + \Delta w^l_j) - F^{kl}_{ij}(w^l_j(t))}{\Delta w^l_j}$
  - $a^{kl-}_{ij}(t) = \lim_{\Delta w^l_j \to 0^-} \dfrac{F^{kl}_{ij}(w^l_j(t) + \Delta w^l_j) - F^{kl}_{ij}(w^l_j(t))}{\Delta w^l_j}$
- Local $A^{(k)}(t)$ can be canonically embedded into the global $A(t)$
- Local $B^{(k)}(t)$ *cannot* be canonically embedded into the global $B(t)$

Elasticity Matrix for Multi Economy

$$A(t) = \left(\begin{array}{c:c:c:c} A^{(1)}(t) & A^{(12)}(t) & \dots & A^{(1s)}(t) \\ \hdashline A^{(21)}(t) & A^{(2)}(t) & & \\ \hdashline \vdots & & \ddots & \\ \hdashline A^{(s1)}(t) & \dots & & A^{(s)}(t) \end{array}\right)$$

- $A^{(kl)}(t) = \left(a_{ij}^{kl}(t)\right)_{\substack{1 \le i \le n_k \\ 1 \le j \le n_l}}$
- Global matrix is canonical embeddedings of local matrices

Jacobian Matrix for Multi Economy

$$B(t) = \left(\begin{array}{c:c:c:c} \tilde{B}^{(1)}(t) & A^{(12)}(t) & \dots & A^{(1s)}(t) \\ \hdashline A^{(21)}(t) & \tilde{B}^{(2)}(t) & & \\ \hdashline \vdots & & \ddots & \\ \hdashline A^{(s1)}(t) & \dots & & \tilde{B}^{(s)}(t) \end{array}\right)$$

- $\tilde{B}^{(k)}(t) \neq B^{(k)}(t)$
- Off-diagonal block matrices $A^{ij}(t)$ $(i \neq j)$ cause contagion

Quantitative Definition of Contagion

- We say that *contagion* in a global economic system occurs if given two times $0 < t_0 < t_1$,
  1. At $t < t_0$, $\max_k \rho\left(B^{(k)}(t)\right) < 1$ and $\rho\left(B(t)\right) < 1$
  2. At $t \in (t_0, t_1)$, $\max_k \rho\left(B^{(k)}(t)\right) > 1$ and $\rho\left(B(t)\right) < 1$
  3. At time $t > t_1$ $B(t) \neq \oplus_{k=1}^{s} B^{(k)}(t)$ and $\rho\left(B(t)\right) > 1$.
- Unrelated simultaneous crises are ruled out:
  - $B(t) = \oplus_{k=1}^{s} B^{(k)}(t)$, then $\rho\left(B(t)\right) = \max_k \rho\left(B^{(k)}(t)\right)$

    $\Longrightarrow$ independent occurrence of sub-systemic crises.
  - Condition 3 implies nonzero off-diagonal block matrices $A^{ij}(t)$ $(i \neq j)$

2010-2011+ Eurozone Crisis I

- Mini Eurozone and Mini Global Economy
  - Group I: Greece (1), Ireland (2), Portugal (3), Spain (4), and Italy (5)
  - Group II: France (6), Germany (7)
  - Group III: USA (8)
- Each economy has 5 agents: C, F, B, G, I (1 - 5)

$$A(t) = \begin{pmatrix} A^{(1)}(t) & \cdots & A^{(16)}(t) & A^{(17)}(t) & 0 \\ \vdots & \ddots & & & \\ A^{(61)}(t) & & A^{(6)}(t) & A^{(67)}(t) & A^{(68)}(t) \\ A^{(71)}(t) & & A^{(76)}(t) & A^{(7)}(t) & A^{(78)}(t) \\ 0 & & A^{(86)}(t) & A^{(87)}(t) & A^{(8)}(t) \end{pmatrix} \quad B(t) = \begin{pmatrix} \tilde{B}^{(1)}(t) & \cdots & A^{(16)}(t) & A^{(17)}(t) & 0 \\ \vdots & \ddots & & & \\ A^{(61)}(t) & & \tilde{B}^{(6)}(t) & A^{(67)}(t) & A^{(68)}(t) \\ A^{(71)}(t) & & A^{(76)}(t) & \tilde{B}^{(7)}(t) & A^{(78)}(t) \\ 0 & & A^{(86)}(t) & A^{(87)}(t) & \tilde{B}^{(8)}(t) \end{pmatrix}$$

2010-2011+ Eurozone Crisis II

Scenario 1. Greek sovereign debt is restructured

- Payments from Greek G to French B ↓: $F^{61}_{34} \downarrow \quad \Rightarrow a^{61}_{34} \downarrow$
  - Entries of $A^{61}(t)$ kept low $\Longrightarrow$ Little impact on $\rho(B(t))$
- Payments from Greek G to German B ↓: $F^{71}_{34} \downarrow \quad \Rightarrow a^{71}_{34} \downarrow$
  - Entries of $A^{71}(t)$ kept low $\Longrightarrow$ Little impact on $\rho(B(t))$
- Fear for French, German banks' insolvency rises
- Markets reduce their exposure to French, German banks
- ECB & Fed's lending to French, German banks ↑
- Post-Lehman Brothers type credit crunch is possible

2010-2011+ Eurozone Crisis III

Scenario 2. Greek sovereign debt is not restructured

- Domestically:
  - French, German banks more susceptible to liquidity crunches
  - $a_{i3}^{6+} \neq a_{i3}^{6-}, \quad a_{i3}^{7+} \neq a_{i3}^{7-}$: hoard cash
- Externally:
  - $a_{33}^{76} \uparrow$ and $a_{33}^{86} \uparrow$: greater default risk of French banks to their German and the US counterparties
  - $a_{33}^{67} \uparrow$ and $a_{33}^{87} \uparrow$: greater default risk of German banks to their French and the US counterparties
  - These belong to the off-diagonal blocks $B(t)$
  - Higher probability for $\rho(B(t)) > 1 \Longrightarrow$ Global financial crisis

2010-2011+ Eurozone Crisis IV

Scenario 3. Fear Factor

- If French B and I lose confidence in Italian sovereign debt:
  - $\mathrm{NSU}^6_{3,t}\left(F^{56}_{43}(t)\right)$ decreases $\Longrightarrow F^{56}_{43}$ decreases
  - $\mathrm{NSU}^6_{5,t}\left(F^{56}_{45}(t)\right)$ decreases $\Longrightarrow F^{56}_{45}$ decreases
- If German B and I lose confidence in Italian sovereign debt:
  - $\mathrm{NSU}^7_{3,t}\left(F^{57}_{43}(t)\right)$ decreases $\Longrightarrow F^{57}_{43}$ decreases
  - $\mathrm{NSU}^7_{5,t}\left(F^{57}_{45}(t)\right)$ decreases $\Longrightarrow F^{57}_{45}$ decreases
- Italian sovereign bond yields soar
- Risk of Italian default rises
- This is not due to *Contagion*

Current Issues

- Scenario 1 does not work well
  - Greek G (now CCC) cannot print euro
  - Austerity deepens recession
- What if Greece is to leave eurozone?
  - Worst: all three scenarios for all major economies
    $\Longrightarrow$ situation grows exponentially worse
  - Hope: depending on exit strategy, things may improve
  - Greek debt: terms of restructuring, then currency control
  - The rest: keep money flow ($\neq$ printing more)
- Group 2: French politics
- Group 3: U.S. banks

1997-98 Asian-Russian Crisis

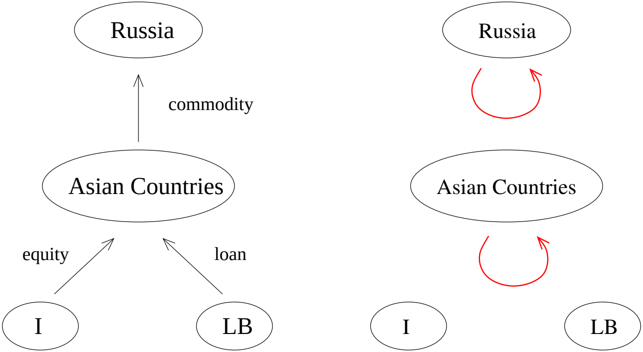


Figure : Flow of funds vs. flow of default among stricken countries and foreign investors

- Each country could devalue its own currency
- Off-block matrices $A^{kl}(t)$ are zero $\Longrightarrow$ no contagion

Conclusion: Work in Progress

- Theoretical
  - Analyze the crisis dynamics
  - Impact of hitting borrowing and liquidity constraints
  - Impact of Government actions: Quantitative easing, taxes, expenditures, bail out, etc.
- Empirical
  - Collect and sort out Flow of Funds data
  - Simulate Instability Indicator
  - Validate the hypothesis that it anticipates systemic crises
  - Simulate Government actions

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