Context: Suppose V is a finite or countable set endowed with some structure that can defined by subsets or ordered sequences of elements, such as:

- a *graph* (V, E) with vertex-set V and edge-set E (as a subset of $V^{\{2\}}$)
- a *map* (V, E, F) with vertex-set V, edge-set E and face-set F
- a *design* (V, B) with point-set V and block-set B (a subset of $2^V = \mathcal{P}(V)$)
- a *polytope* $(\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n)$ with $\mathcal{F}_i$ as the set of all i-faces (the i-subflags of the flag-set V).

Symmetry and automorphisms: Generally, an object is said to have symmetry if it can be *transformed in way that leaves it looking the same as it did originally.* The degree of symmetry of a discrete structure $\mathcal{D}$ can be described or measured by its *automorphisms*, which are incidence-preserving bijections from $\mathcal{D}$ to $\mathcal{D}$. Under composition, these form a group, called the *automorphism group* (or *symmetry group*) of the structure $\mathcal{D}$, and this is denoted by $\mathrm{Aut}\,\mathcal{D}$.

Special case: An *automorphism* (or *symmetry*) of a simple **graph** $X = (V, E)$ is a permutation of the vertices of X which preserves the relation of adjacency; that is, a bijection $\pi : V \to V$ such that $\{v^\pi, w^\pi\} \in E$ iff $\{v, w\} \in E$. The automorphism group of X is denoted by $\mathrm{Aut}\,X$.

Examples
(a) Complete graphs and null graphs: $\mathrm{Aut}\,K_n \cong \mathrm{Aut}\,N_n \cong S_n$ for all n
(b) Simple cycles: $\mathrm{Aut}\,C_n \cong D_n$ for all $n \geq 3$
(c) Petersen graph: $\mathrm{Aut}\,P \cong S_5$.

Exercise: How many automorphisms has the underlying graph (1-skeleton) of each of the five Platonic solids: the regular tetrahedron, cube, octahedron, dodecahedron and icosahedron?

Digression: the Degree-Diameter problem

Suppose X is a d-regular simple graph of diameter D. Counting the largest possible number of vertices at distance k from a given vertex, for $1 \leq k \leq D$, gives the **Moore bound**

$$|V(X)| \leq 1 + d + d(d-1) + d(d-1)^2 + \cdots + d(d-1)^{D-1}.$$

This bound is attained for certain pairs (d, D), but certainly not all. The *Degree-Diameter problem* is to find for a given pair (d, D) the largest possible d-regular graph of diameter D. Spectacular progress on this problem was made recently by PhD student Eyal Loz (2005–08); see `http://moorebound.indstate.edu/wiki/The_Degree_Diameter_Problem_for_General_Graphs`.

Transitivity: A graph $X = (V, E)$ is said to be

- *vertex-transitive* if $\mathrm{Aut}\,X$ has a single orbit on the vertex-set V
- *edge-transitive* if $\mathrm{Aut}\,X$ has a single orbit on the edge-set E
- *arc-transitive* if $\mathrm{Aut}\,X$ has a single orbit on the arc-set $A = \{(v, w) : \{v, w\} \in E\}$ of X.

Note that every vertex-transitive graph is regular (in the sense of having all vertices of the same degree/valency). Arc-transitive graphs are also known as *symmetric* graphs.

Exercise: Let X be a k-valent graph, where k is odd. Show that if X is both vertex-transitive and edge-transitive, then also X is arc-transitive. [Harder question: Does the same thing always happen when k is even?]

Another kind of transitivity: An s-**arc** in a graph $X = (V, E)$ is a sequence $(v_0, v_1, \ldots, v_s)$ of vertices of X in which any two consecutive vertices are adjacent and any three consecutive vertices are distinct, that is, $\{v_{i-1}, v_i\} \in E$ for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i < s$. The graph $X = (V, E)$ is called *s-arc-transitive* if $\mathrm{Aut}\, X$ is transitive on the set of all s-arcs in X.

Examples

(a) Complete graphs: K_n is 2-arc-transitive (but not 3-arc-transitive) for all $n \ge 3$
(b) Simple cycles: C_n is s-arc-transitive for all $s \ge 0$, whenever $n \ge 3$
(c) The Petersen graph is 3-arc-transitive (but not 4-arc-transitive)
(d) The Heawood graph is 4-arc-transitive (but not 5-arc-transitive)
(e) Tutte's 8-cage is 5-arc-transitive (but not 6-arc-transitive).

Exercise: For each of the five Platonic solids, what is the largest value of s such that the underlying graph (1-skeleton) is s-arc-transitive?

Exercise: Let X be an s-arc-transitive d-valent connected simple graph. Find a lower bound on the order of the stabilizer in $\mathrm{Aut}\, X$ of a vertex $v \in V(X)$, in terms of s and d.

Reflexibility and Chirality

If an object is equivalent to its mirror image (with respect to some axis/hyperplane) then it is said to have *reflectional symmetry*, or be *reflexible*. For example, all of the Platonic solids are reflexible, as is a perfect snowflake.

On the other hand, an object is called *chiral* if it differs from all of its mirror images. For example, the pattern of seeds on a sunflower is chiral. Also the right and left trefoil knots are chiral, each being a mirror image of the other.

[In fact, the term 'chiral' means *handedness*, derived from the Greek word $\chi\epsilon\iota\rho$ (or 'kheir') for 'hand'. It is usually attributed to the scientist William Thomson (Lord Kelvin) in 1884, although the philosopher Kant had earlier made the observation that left and right hands are inequivalent except under mirror image.]

Question: Among objects occurring frequently in nature that have some degree of symmetry — that is, more than one automorphism — how many are reflexible, and how many are chiral?

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Definition: A *map* M is an embedding of a connected graph or multigraph X into a surface S, with the property that all of the components of $S \setminus X$ (obtained by removing X from S) are homeomorphic to unit disks — called the *faces* of the map. The map is *orientable* or *non-orientable* depending on whether it lies on an orientable surface (such as the sphere, torus, or double torus) or a non-orientable surface (such as the real projective plane or the Klein bottle).

Automorphisms of maps: An *automorphism* of a map M is an incidence-preserving permutation of each of the vertex-set $V = V(M)$, the edge-set $E = E(M)$ and the face-set $F = F(M)$.

Lemma: Any automorphism of a map M is completely determined by its effect on a given *flag* (incident vertex-edge-face triple). Hence $|\mathrm{Aut}\, M| \le 4|E|$.

Definitions: If $|\mathrm{Aut}\, M| = 4|E|$, then the action of $\mathrm{Aut}\, M$ is regular (sharply-transitive) on flags of M, in which case M is called a *regular map*. A weaker version of this applies to orientable maps: if M is orientable, and the group $\mathrm{Aut}^{\mathrm{o}} M$ of all orientation-preserving automorphisms of M acts regularly on the *arcs* of M, then M is called *orientably-regular* (or *rotary*, or *regular*).

Note the ambiguity! In fact there are three kinds of rotary/regular maps:
- Orientable maps M with $|\mathrm{Aut}\, M| = 4|E|$ ... these are *reflexible* (admitting reflections);
- Orientable maps M with $|\mathrm{Aut}\, M| = |\mathrm{Aut}^{\mathrm{o}} M| = 2|E|$ ... these are *chiral* (or irreflexible);
- Non-orientable maps M with $|\mathrm{Aut}\, M| = 4|E|$.

The maps in the first and third cases are *flag-transitive*, and sometimes called *fully regular*.

Type: In each of the above cases, $\mathrm{Aut}\, M$ is transitive on vertices, on edges and on faces of M. It follows that the underlying graph X is regular, with *valence* m, say, and that every face of M has the same size k, called the *co-valence* of M. The pair $\{k, m\}$ is called the *type* of M.

Examples: The five Platonic solids may be viewed as (fully) regular maps on the sphere, of types $\{3, 3\}$ (tetrahedron), $\{3, 4\}$ (octahedron), $\{4, 3\}$ (cube), $\{3, 5\}$ (icosahedron), $\{5, 3\}$ (dodecahedron). On the torus are infinitely many regular maps of type $\{3, 6\}$ (triangulations), type $\{4, 4\}$ (quadrangulations), and type $\{6, 3\}$ (honeycombs).

Exercise: Find the numbers of vertices, edges and faces of such maps (on the sphere and torus), and the orders of their automorphism groups.

Genus formulae: If M is an orientably-regular map, with $|V|$ vertices, $|E|$ edges and $|F|$ faces, then by arc-transitivity, $k|V| = 2|E| = m|F| = |\mathrm{Aut}^{\mathrm{o}} M|$, and so by the Euler-Poincaré formula, the characteristic χ and genus g of the surface (and the map) are given by

$$2 - 2g = \chi = |V| - |E| + |F| = |\mathrm{Aut}^{\mathrm{o}} M|\,(\frac{1}{k} - \frac{1}{2} + \frac{1}{m}).$$

Similarly, if M is flag-transitive, then $2k|V| = 4|E| = 2m|F| = |\mathrm{Aut}\, M|$, and then

$$|\mathrm{Aut}\, M|\,(\frac{1}{2k} - \frac{1}{4} + \frac{1}{2m}) = |V| - |E| + |F| = \chi = \begin{cases} 2 - 2g & \text{if } M \text{ is orientable} \\ 2 - g & \text{if } M \text{ is nonorientable.} \end{cases}$$

Exercises:
(a) Find all pairs (m, k) such that $\frac{1}{k} - \frac{1}{2} + \frac{1}{m} > 0$, and all (m, k) such that $\frac{1}{k} - \frac{1}{2} + \frac{1}{m} = 0$.
(b) Show that if $\frac{1}{k} - \frac{1}{2} + \frac{1}{m} < 0$, then $|\frac{1}{k} - \frac{1}{2} + \frac{1}{m}| \ge \frac{1}{42}$, so that $|\mathrm{Aut}\, M| \le -84\chi$.
(c) Find a list of potential candidates for m, k and $|\mathrm{Aut}\, M|$ for the case $\chi = -2$.

Generators for Aut M: If M is a rotary/regular map of type $\{m,k\}$, then for any flag (v,e,f) there exist automorphisms R and S such that R cyclically permutes consecutive edges of the face f, and S cyclically permutes consecutive edges incident to v, and RS reverses the edge e. These elements R and S generate an arc-transitive subgroup of $\mathrm{Aut}\,M$, of order $2|E|$ if M is orientable, or $4|E|$ if M is non-orientable, and they satisfy the relations $R^m = S^k = (RS)^2 = 1$.
[In fact, Steve Wilson likes to use a refinement of this property to *define* a rotary map.]
When M is flag-transitive, there are automorphisms a,b,c of order 2 such that $(ab)^m = (bc)^k = (ac)^2 = 1$, with $ab = R$ and $bc = S$, and these involutions generate $\mathrm{Aut}\,M$, of order $4|E|$.

Connection with triangle groups: If M is an orientably-regular map of type $\{m,k\}$, then $\mathrm{Aut}^{\mathrm{o}}M$ is a quotient of $\Delta^{\mathrm{o}}(m,k,2) = \langle\, x,y,z \mid x^m = y^k = z^2 = xyz = 1\,\rangle$, the *ordinary* $(m,k,2)$ *triangle group.* Similarly if M is a flag-transitive map of type $\{m,k\}$, then $\mathrm{Aut}\,M$ is a quotient of $\Delta(m,k,2) = \langle\, a,b,c \mid a^2 = b^2 = c^2 = (ab)^m = (bc)^k = (ac)^2 = 1\,\rangle$, the *full* $(m,k,2)$ *triangle group.* Conversely, if G is a finite homomorphic image of one of these groups, in which the orders of the key elements are preserved, then G is an arc-transitive (respectively flag-transitive) group of automorphisms of a regular map M that can be constructed from the homorphism onto G. Vertices, edges and faces of M can be labelled by (right) cosets in G of $\langle y\rangle$, $\langle z\rangle$ and $\langle x\rangle$, or $\langle b,c\rangle$, $\langle a,c\rangle$ and $\langle a,b\rangle$, respectively, with incidence given by non-empty intersection.

Reflexibility: If M is an orientably-regular map of type $\{m,k\}$, with $\mathrm{Aut}^{\mathrm{o}}M$ generated by R and S s.t. $R^m = S^k = (RS)^2 = 1$, as above, then M *is reflexible if and only if* $\mathrm{Aut}^{\mathrm{o}}M$ *has an involutory automorphism* β *such that* $R^\beta = R^{-1}$ *and* $S^\beta = S^{-1}$.

Orientability: If M is a flag-transitive map of type $\{m,k\}$, with $\mathrm{Aut}\,M$ generated by a,b,c of order 2 s.t. $(ab)^m = (bc)^k = (ac)^2 = 1$, as above, then M *is orientable if and only if the subgroup generated by* $R = ab$ *and* $S = bc$ *has index* 2 *in* $\mathrm{Aut}\,M$.

Duality: The *geometric/topological dual* of a map M is obtained by interchanging the roles of vertices and faces, preserving incidence with edges. The dual M^* of a rotary/regular map of type $\{m,k\}$ has type $\{k,m\}$. For an orientably-regular map M as above, this is achieved by the correspondence $R \leftrightarrow S$, or for a flag-transitive map M, by $(a,b,c) \leftrightarrow (c^b, b, a^b)$.
[Hence M^* is isomorphic to the mirror image of the map obtained from the polytope dual correspondence $(R,S) \leftrightarrow (S^{-1}, R^{-1})$, which comes from flag reversal $(a,b,c) \leftrightarrow (c,b,a)$.]

Finding regular maps of given small genera: By the connection with triangle groups and the Euler-Poincaré (genus) formula, finding all regular maps of given Euler characteristic χ reduces to finding all smooth quotients of relevant triangle groups of particular orders. This can be done using algebra and computation, to build up a *census of examples, which is now complete for genus* 2 *to* 300 [MC, 2011]. Pictures of some of the maps have been drawn very nicely by Jarke van Wijk `http://www.win.tue.nl/~vanwijk/regularmaps`.

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Preamble: In the study of maps and polytopes (and other discrete structures) with a high degree of symmetry, frequently *the most symmetric examples* fall into classes with the property that the automorphism group of every member of the class is a quotient of some *universal group*. The following ideas and techniques can be very helpful in dealing with such families.

Schreier coset graphs: Let G be a group generated by a finite set $X = \{x_1, x_2, \ldots, x_d\}$. Given any *transitive permutation representation* of G on a finite set Ω, we may form a graph with vertex-set Ω, and edges of the form $\alpha — \alpha x_i$ for $1 \le i \le d$. Similarly, if H is a *subgroup of finite index* in G, we may form a graph whose vertices are the right cosets of H and edges are of the form $Hg — Hgx_i$ for $1 \le i \le d$. These two graphs are the same when Ω is the coset space $(G\!:\!H)$, or H is the stabilizer of a point of Ω. It is called the *Schreier coset graph* $\Sigma(G, X, H)$. This gives a *diagrammatic representation* of the natural action of G on cosets of H.

Coset tables: The natural action of G on cosets of H can also be given by a *coset table*, with rows indexed by right cosets of H, columns indexed by elements of X and their inverses, and the entry in row Hg and column x_i (resp. x_i^{-1}) representing the coset Hgx_i (resp. Hgx_i^{-1}).

Exercise: Use coset graphs or coset tables to find all transitive permutation representations of the modular group $C_2 * C_3 = \langle\, x, y \mid x^2 = y^3 = 1\,\rangle$ on up to 6 points.

Reidemeister-Schreier process: Given a subgroup H of finite index in a finitely-presented group G, it is possible to find a presentation for H (with finitely many generators and relations). A *Schreier transversal* T for H in G corresponds to a spanning tree for the coset graph Σ, and *Schreier generators* for H in G correspond to edges of the coset graph not used in the spanning tree. Defining relations for H (in terms of the Schreier generators) can then be found by tracing the relations for G around the coset graph, from each vertex in turn.

Ree-Singerman theorem: Let G be a transitive permutation group, generated by elements $x_1, x_2, \ldots, x_d$ such that $x_1 x_2 \ldots x_d = 1$ (identity), let n be the degree $|\Omega|$, and let c_i be the number of orbits of $\langle x_i \rangle$ on Ω. Then $c_1 + c_2 + \cdots + c_d \le (d-2)n + 2$.

Other uses of coset graphs: Schreier coset can be simplified to 'coset diagrams' (e.g. by deleting vertex and edge labels, deleting loops for fixed points of generators, using single edges for 2-cycles of involutory generators, and ignoring the effect of redundant generators). Often two coset diagrams for the same group G on (say) m and n points can be *composed* to produce a transitive permutation representation of larger degree $m+n$. In turn, this method may be used to construct *abelian covers*, or infinite families of quotients of various kinds (e.g. automorphim groups of chiral maps), or prove that a given finitely-presented group is infinite,

Coset enumeration: Given a finitely-presented group $G = \langle\, X \mid R\,\rangle$ and a subgroup H generated by a given finite set Y of words on the alphabet $X = \{x_1, \ldots, x_m\}$, methods exist for systematically enumerating the cosets Hg for $g \in G$, by using the generators and relations to help construct the coset table. Each relator $r \in R$ from the defining presentation forces pairs of cosets to be equal, and the same thing happens on application of each generator $y \in Y$ to the trivial coset H. New cosets are defined (if needed), and all such coincidences are processed, until the coset table either 'closes' or has too many rows. If the coset table *closes* with n cosets, then $|G\!:\!H| = n$, and the table gives the natural permutation representation of G on the coset space $(G\!:\!H)$. If it does not close, then the index $|G\!:\!H|$ could be infinite, or just too large to be found (or it might even be small but the computation was not given enough resources).

Finding small index subgroups: Given a finitely-presented group $G = \langle X \mid R \rangle$ and a positive integer n, it is possible to find all subgroups of index up to n in G (up to conjugacy) by systematic enumeration of all coset tables with at most n rows. This method uses a backtrack process, which starts by taking H as the identity subgroup and attempts to construct the coset table for H in G. At each stage of the process, if more than n cosets of H are defined, then coincidences are forced between them, which has the effect of including new elements in H (since $Ha = Hb$ if and only if $ab^{-1} \in H$). Often such a coincidence will be seen to lead to a subgroup H conjugate to one found previously, in which case that coincidence is rejected. If not rejected, then ab^{-1} is added to a (partial) set of generators for H, and the search continues.
[Note: This process will terminate (given sufficient time and memory), by Schreier's theorem: every subgroup of finite index in a finitely-generated group is itself finitely-generated.]

Finding small index *normal* subgroups: Small homomorphic images of a finitely-presented group G can be found as the groups of permutations induced by G on cosets of subgroups of small index. This gives G/K where K is the core of H, but produces only images that have small degree faithful permutation representations. A new method was developed recently by Derek Holt and his student David Firth, which systematically enumerates all possibilities for a composition series of a factor group G/K, where K is a normal subgroup of small index in G.

Exercise: Find all normal subgroups of index up to 12 in $\langle\, x, y \mid x^2 = y^3 = 1 \,\rangle$.

Double-coset graphs: Let G be a group, H a subgroup of G, and a an element of G such that $a^2 \in H$. Now define a graph $\Gamma = \Gamma(G, H, a)$ by taking right cosets Hg (for $g \in G$) as vertices, and joining Hx to Hy by an edge whenever xy^{-1} lies in the double coset HaH. This graph is connected if and only if HaH generates G. More importantly, right multiplication makes G a vertex-transitive group of automorphisms of Γ, with the subgroup H stabilizing the vertex H and acting transitively on its neighbours Hah (for $h \in H$). Hence Γ is arc-transitive.
This gives a powerful construction for families of symmetric graphs, and has been used to provide a *census of all symmetric* 3*-valent graphs on up to* 10,000 *vertices* [MC, 2011].

Computer implementations: Many of the above methods have been implemented in the computational algebra systems MAGMA and GAP. For example, MAGMA has these commands:

- `CosetTable` and `ToddCoxeter` (for coset enumeration)
- `Rewrite` (for the Reidemeister-Schreier process)
- `LowIndexSubgroups` and `LowIndexNormalSubgroups` (for finding small index subgroups).

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Classification of regular maps: Regular maps can be viewed and categorised by several perspectives, e.g. by *type*, by *surface/genus*, by *underlying graph*, or by *automorphism group*. Within each class, various properties are worth considering, e.g. orientability, reflexibility, duality, Petrie duality, etc., or having *simple* underlying graph (in the map and/or its dual), or having a *Cayley graph* as underlying graph, or existence of *semi-regular automorphisms*, or the kinds of *quotients* and *covers* that each map admits.

Classification by type of the map: In this case, relatively little is known. Regular maps of types $\{2, m\}$, $\{3, 3\}$, $\{3, 4\}$ and $\{3, 5\}$ (on the *sphere*) and types $\{3, 6\}$ and $\{4, 4\}$ (on the *torus*) and their duals all known and well understood. It is also known that there are *infinitely many rotary maps of any given hyperbolic type* $\{k, m\}$; this follows easily from the residual finiteness of the triangle group $\Delta^o(2, k, m)$. Quite a lot is known about *Hurwitz maps*, which are regular maps of type $\{3, 7\}$. Similarly, constructions are known for *infinite families of rotary but chiral maps* of various given types, e.g. type $\{3, m\}$ for all $m \geq 7$.

Exercise: For some pair (k, m) with $\frac{1}{k} + \frac{1}{m} < \frac{1}{2}$, find a specific construction for infinitely many regular maps of type $\{m, k\}$. [This might be difficult, so here is a hint: consider semi-direct products $C_n \rtimes H$ of a cyclic group of variable order by a group H of fixed order.]

Classification by automorphism group: Again here, relatively little has been achieved. The classifications for cyclic groups and dihedral groups are straightforward exercises (below). Classifications of regular maps M with $\mathrm{Aut}^o M \cong \mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$ follow from work by Macbeath (1967), Sah (1969), and Conder, Potočnik and Sirá̌n (2009). Some partial classifications are also known for the alternating and symmetric groups (through the study of actions of these groups on Riemann surfaces). Also in the flag-transitive (fully regular) case, many sub-classifications follow from the study of groups generated by three involutions, two of which commute; see papers by Conder (1980s), Sjerve & Cherkassoff (1994), Nuzhin (1996–), Tamburini & Zucca (1997), and Mazurov (2003).

Exercise: For any integer $n > 2$, find all possible ways of generating the cyclic group C_{2n} and the dihedral group D_n (of order $2n$) by (a) two elements R and S such that RS has order 2, and (b) three elements a, b, c of order 2 such that ac has order 2.

Exercise: Find all possible ways of generating the simple group A_5 by (a) two elements R and S such that RS has order 2, and (b) three elements a, b, c of order 2 such that ac has order 2.

Classification by underlying graph: Much more progress has been made on this topic. Embeddings of the following families of graphs as orientably regular maps are now known: cycle graphs C_n (on sphere); complete graphs K_n [James & Jones (1985)]; cocktail party graphs [Nedela & Škoviera (1996)]; merged Johnson graphs [Jones (2005)]; some complete multipartite graphs [Du et al (2005)] ; complete bipartite graphs $K_{n,n}$ [various authors]; graphs of order p or pq where p and q are prime [various authors]; hypercube graphs Q_n for n odd [Du et al (2007)] and for n even [Catalano et al (2010)]; Hamming graphs $H_n(q)$ for $q > 2$ [Jones (preprint)].

Exercise: In what ways can the 3-cube Q_3 be embedded in a surface as a regular map, apart from the standard embedding on the sphere?

Exercise: In what ways can the complete bipartite graph $K_{3,3}$ be embedded in a surface as a regular map?

Classification by surface (genus/characteristic):

This is perhaps the most illuminating perspective. Regular maps of characteristic 0, 1 and 2 were classified by Brahana (1927) and Coxeter (1957). Orientably-regular maps of genus 3 were classified by Sherk (1959), and those of genus 4, 5 and 6 by Garbe (1969). With the help of computational methods, most recently the `LowIndexNormalSubgroups` facility in MAGMA, a complete list of all orientable and non-orientable regular maps is now known, for genus 2 to 300.

Theoretical analyses have determined all non-orientable regular maps of genus $p+2$ for p an odd prime [Breda, Nedela and Siráň (2005)], and moreover, all rotary maps M for which the order of the subgroup generated by the canonical automorphisms R and S is coprime to the characteristic χ when χ is odd, or to $\chi/2$ when χ is even [Conder, Siráň & Tucker (2010)]. These determinations have resulted in the following theorems:

• If M is an irreflexible (chiral) orientably-regular map of genus $p+1$ where p is prime, then $p \equiv 1 \bmod 3$ and M has type $\{6,6\}$, or $p \equiv 1 \bmod 5$ and M has type $\{5,10\}$, or $p \equiv 1 \bmod 8$ and M has type $\{8,8\}$. In particular, *there are no chiral maps of genus* $p+1$ *whenever* p *is a prime such that* $p-1$ *is not divisible by* 3, 5 *or* 8.

• There is *no reflexible regular map with simple underlying graph* on an orientable surface of genus $p+1$ where p is a prime congruent to 1 mod 6, for $p > 13$.

• There is no *regular map at all* on a non-orientable surface of genus $p+2$ where p is a prime congruent to 1 mod 12, for $p > 13$.

Regular Cayley maps: If the underlying graph of the rotary/regular map M is a *Cayley graph* $\mathrm{Cay}(G,S)$ for some group G, such that G acts regularly on the vertices of M (as a group of map automorphisms), then M is a regular Cayley map for G. Quite a lot of recent research has been done on regular Cayley maps (RCMs), including: general theory [Jajcay, Siráň, et al]; *balanced* regular Cayley maps for cyclic, dihedral and generalized quaternion groups [Wang & Feng]; RCMs for *abelian* groups [MC, Jajcay & Tucker]; RCMs *of prime valency* for abelian, dihedral and dicyclic groups [Kim, Kwon & Lee]; *t-balanced* RCMs for cyclic groups [Kwon] and semi-dihedral groups [Oh]; RCMs for *dihedral* groups [Kovács]; and a complete determination of *all regular Cayley maps for finite cyclic groups* [Conder & Tucker (preprint)].

Additional symmetries: Very recent research has considered regular maps that admit a high degree of 'external symmetry', including duality and Petrie duality, and the (Wilson) 'hole' operators. The best of these maps are said to be *kaleidoscopic maps with trinity symmetry.*

A (small) selection of references:

D. Archdeacon, M.D.E. Conder and J. Širáň, Trinity symmetry and kaleidoscopic regular maps, preprint.

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Ya. N. Nuzhin, Generating triples of involutions of Lie-type groups over a finite field of odd characteristic, II, *Algebra and Logic* 36 (1997), 245–256.

Definition(s): An *abstract polytope* of rank n is a partially ordered set $\mathcal{P}$ endowed with a strictly monotone rank function having range $\{-1,\ldots,n\}$. For $-1 \le j \le n$, elements of $\mathcal{P}$ of rank j are called the *j-faces*, and a typical j-face is denoted by F_j. This poset $\mathcal{P}$ has a smallest (-1)-face F_{-1}, and a greatest n-face F_n, and each maximal chain (or *flag*) of $\mathcal{P}$ has length $n+2$. The faces of rank $0, 1$ and $n-1$ are called the *vertices, edges* and *facets* of the polytope, respectively. Two flags are called *adjacent* if they differ by just one face. The poset $\mathcal{P}$ must be *strongly flag-connected*, which means that any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$, such that each two successive faces Φ_{i-1} and Φ_i are adjacent, and $\Phi \cap \Psi \subseteq \Phi_i$ for all i. Finally, whenever $F \le G$, with $\mathrm{rank}(F) = j-1$ and $\mathrm{rank}(G) = j+1$, there are *exactly two* faces H of rank j such that $F \le H \le G$. The latter is called the *diamond condition*.

Symmetries of abstract polytopes: An *automorphism* of an abstract polytope $\mathcal{P}$ is an order-preserving bijection $\mathcal{P} \to \mathcal{P}$.

Exercise: Use the diamond condition and strongly flag-connectivity to to prove that *every automorphism of an abstract polytope is uniquely determined by its effect on any given flag.*

Regular polytopes: An abstract polytope $\mathcal{P}$ is *regular* if its automorphism group $\mathrm{Aut}\,\mathcal{P}$ is transitive (and hence regular) on the flags of $\mathcal{P}$.

The automorphism group of a regular polytope: Let $\mathcal{P}$ be a regular abstract polytope, and let Φ be any flag $F_{-1} - F_0 - F_1 - F_2 - \ldots - F_{n-1} - F_n$. Call this the *base flag.* For $0 \le i \le n$–1, there is an involutory automorphism ρ_i that maps Φ to the adjacent flag Φ^i (which differs from Φ only in its i-face). Let k_i be the order of the product $\sigma_i = \rho_{i-1}\rho_i$, for $1 \le i \le n$–1. Then these automorphisms $\rho_0, \rho_1, \ldots, \rho_{n-1}$ generate $\mathrm{Aut}\,\mathcal{P}$, and satisfy the following relations: $\rho_i^2 = 1$ for $0 \le i \le n-1$, $(\rho_{i-1}\rho_i)^{k_i} = 1$ for $1 \le i \le n-1$, and $(\rho_i\rho_j)^2 = 1$ for $0 \le i < i+1 < j \le n-1$. These are precisely the defining relations for the *Coxeter group* $[\,k_1, k_2, .., k_{n-1}]$ (with Schläfli symbol $\{\,k_1 \,|\, k_2 \,|\, .. \,|\, k_{n-1}\}$). In particular, $\mathrm{Aut}\,\mathcal{P}$ is a quotient of this Coxeter group.

Stabilizers and cosets: For $0 \le i \le n-1$, the i-faces of a regular polytope $\mathcal{P}$ can be identified with the (right) cosets of the subgroup generated by $\rho_0, \rho_1, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_{n-1}$. Then incidence in $\mathcal{P}$ is given by non-empty intersection of cosets.

The Intersection Condition: When $\mathcal{P}$ is regular, the generators ρ_i for $\mathrm{Aut}\,\mathcal{P}$ satisfy the condition $\langle\, \rho_i : i \in I \,\rangle \,\cap\, \langle\, \rho_i : i \in J \,\rangle = \langle\, \rho_i : i \in I \cap J \,\rangle$ for all $I, J \subseteq \{0, 1, \ldots, n-1\}$.

Conversely, this condition on generators $\rho_0, \rho_1, \ldots, \rho_{n-1}$ of a smooth quotient of a Coxeter group $[\,k_1, k_2, .., k_{n-1}]$ ensures the diamond condition and strong flag connectivity, and therefore:

If G is a finite group generated by n elements $\rho_0, \rho_1, \ldots, \rho_{n-1}$ which satisfy the defining relations for a string Coxeter group of rank n and satisfy the intersection condition, then there exists a polytope $\mathcal{P}$ with $\mathrm{Aut}\,\mathcal{P} \cong G$.

Infinite families of regular polytopes: There are many such families, including these:

- *Regular n-simplex*, of type $[\,3, \overset{n-1}{\ldots}, 3\,]$, with automorphism group S_n
- *Cross polytope* (or *n-orthoplex*), of type $[\,3, \overset{n-2}{\ldots}, 3, 4\,]$
- *n-dimensional cubic honeycomb*, of type $[\,4, 3, \overset{n-2}{\ldots}, 3, 4\,]$.

The 'rotation subgroup' of a regular polytope: In $\operatorname{Aut}\mathcal{P} = \langle \rho_0, \rho_1, \ldots, \rho_{n-2}, \rho_{n-1}\rangle$, we may define $\sigma_j = \rho_{j-1}\rho_j$ for $1 \le j \le n-1$. These generate a subgroup of index 1 or 2 in $\operatorname{Aut}\mathcal{P}$, containing denoted by $\operatorname{Aut}^+\mathcal{P}$, or $\operatorname{Aut}^o\mathcal{P}$. If the index is 1, then $\operatorname{Aut}^+\mathcal{P} = \operatorname{Aut}\mathcal{P}$ has a single orbit on flags of $\mathcal{P}$, but if the index is 2, then $\operatorname{Aut}^+\mathcal{P}$ has two orbits on flags, with adjacent flags in different orbits.

Exercise: Prove that ${\sigma_1}^{\rho_0} = {\sigma_1}^{-1}$, and ${\sigma_2}^{\rho_0} = {\sigma_1}^2\sigma_2$, while ${\sigma_i}^{\rho_0} = \sigma_i$ for $3 \le i \le n-1$.

Chirality: An abstract n-polytope $\mathcal{P}$ is *chiral* if its automorphism group has two orbits on flags, with adjacent flags being in distinct orbits. In this case, for each flag $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$, there are automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ such that each σ_j fixes all faces in $\Phi \setminus \{F_{j-1}, F_j\}$, and cyclically permutes j-faces in the rank 2 section $[F_{j-2}, F_{j+1}] = \{\, F \in \mathcal{P} \mid F_{j-2} \le F \le F_{j+1}\}$. These automorphisms generate $\operatorname{Aut}\mathcal{P}$), and satisfy (among others) the relations

$$(\sigma_i\sigma_{i+1}\ldots\sigma_j)^2 = 1 \quad \text{for} \ \ 1 \le i < j \le n-1,$$

which are defining relations for the orientation-preserving subgroup of the Coxeter group $[k_1, \ldots, k_{n-1}]$, namely the subgroup generated by the elements $\sigma_i = \rho_{i-1}\rho_i$ for $1 \le i < n$.

Conversely, *if G is any finite group generated by elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ satisfying these relations and a modified version of the intersection condition, then there exists an abstract polytope $\mathcal{P}$ of rank n which is regular or chiral, with $G \cong \operatorname{Aut}\mathcal{P}$ if $\mathcal{P}$ is chiral, or $G \cong \operatorname{Aut}^+\mathcal{P}$ of index 2 in $\operatorname{Aut}\mathcal{P}$ if $\mathcal{P}$ is regular.*

Moreover, *the polytope $\mathcal{P}$ is* regular *if and only if there exists an involutory group automorphism $\rho\colon \operatorname{Aut}\mathcal{P} \to \operatorname{Aut}\mathcal{P}$ such that $\rho(\sigma_1) = {\sigma_1}^{-1}$, $\rho(\sigma_2) = {\sigma_1}^2\sigma_2$, and $\rho(\sigma_i) = \sigma_i$ for $3 \le i \le n-1$* (or in other words, acting like conjugation by the generator ρ_0 in the regular case).

Chiral polytopes (for which no such additional automorphism exists) occur in pairs of enantiomorphic forms, with one being the 'mirror image' of the other.

Duality: The *dual* of an n-polytope $\mathcal{P}$ is the n-polytope $\mathcal{P}^*$ obtained from $\mathcal{P}$ by reversing the partial order. The polytope $\mathcal{P}$ is called *self-dual* if $\mathcal{P} \cong \mathcal{P}^*$. In that case an incidence-reversing bijection $\delta\colon \mathcal{P} \to \mathcal{P}$ is called a *duality*. If $\mathcal{P}$ is a chiral n-polytope, the reverse of a flag can lie in either one of two flag orbits. We say that $\mathcal{P}$ is *properly self-dual* if there exists a duality of $\mathcal{P}$ mapping a flag Φ to a flag Φ^δ in the *same orbit* as Φ (under $\operatorname{Aut}\mathcal{P}$), or *improperly self-dual* if $\mathcal{P}$ has a duality mapping the flag Φ to a flag in the *other orbit* of $\operatorname{Aut}\mathcal{P}$. For 3-polytopes considered as maps, the polytope dual is a mirror image of the map dual. Hence the map is self-dual (as a map) if and only if it is improperly self-dual as a 3-polytope.

Finding chiral polytopes: Chiral polytopes *appear to be much more rare* than regular polytopes, which is surprising since they have a smaller degree of symmetry. This may just hold for small examples, or for small ranks, or of course it could be simply that we don't know enough examples! Chiral polytopes can be *constructed from string Coxeter groups, or by using other algebraic/combinatorial/geometric methods* (e.g. building 'new' ones from old).

Drawback to inductive construction(s): If $\mathcal{P}$ is a chiral n-polytope, then the stabilizer in $\operatorname{Aut}\mathcal{P}$ of each $(n-2)$-face F_{n-2} of $\mathcal{P}$ is transitive on the flags of F_{n-2}, and therefore *every $(n-2)$-face of $\mathcal{P}$ is regular*. Hence there is no natural construction for building chiral polytopes from each rank to the next.

Some references

H.S.M. Coxeter, *Regular Polytopes*, 3rd ed., Dover, New York (1973).

P. McMullen and E. Schulte, *Abstract Regular Polytopes*, Encyclopedia of Mathematics and its Applications 92, Cambridge (2002).

Constructions for regular polytopes:

• C-group permutation representation graphs (CPR graphs) were used by Daniel Pellicer to construct regular polyhedra with alternating groups A_n as automorphism groups (2008), and regular polytopes with given facets and prescribed even last entry of the Schläfli symbol (2010).

• Egon Schulte and Peter McMullen (2002) introduced a new group-theoretic method for constructing a new regular polytope from two given regular polytopes $\mathcal{P}$ and $\mathcal{Q}$, called the 'mix' of $\mathcal{P}$ and $\mathcal{Q}$.

• Polytopes of given type: Dimitri Leemans and Michael Hartley (2004, 2009) constructed various regular 4-polytopes with type $[5, 3, 5]$. Similarly, many families of examples (of type $[3, 5, 3]$, for example) arise from quotients of groups associated with hyperbolic 3-manifolds of small volume, found by Lorimer, Jones, Conder, Torstensson et al (1990s–).

• Amalgamation of polytopes: Michael Hartley constructed regular polytopes with given facets and given vertex-figures, in some special cases (2010).

Collecting small examples of regular polytopes

• Michael Hartley has created a web-based atlas of regular polytopes with automorphism group of order at most 2000, except those ones with automorphism group having order 1024 or 1536 — see `http://www.abstract-polytopes.com/atlas` for this.

• Dimitri Leemans and Laurence Vauthier have found all regular polytopes with automorphism group G being an almost simple group with $S \leq G \leq \mathrm{Aut}(S)$ for some simple group S of order less than 900,000. For the complete list, see `http://cso.ulb.ac.be/∼dleemans/polytopes`.

Both of these two atlases were first published in 2006.

Regular polytopes with given group:

• Dimitri Leemans and Laurence Vauthier proved (in 2006) that the group $\mathrm{PSL}(2, q)$ cannot be the automorphism group of a regular n-polytope for any $n \geq 5$.

• Dimitri Leemans and Egon Schulte determined all regular 4-polytopes with automorphism group $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$, in 2007 and 2009.

• Daniel Pellicer (2008) used CPR graphs to construct regular polyhedra with automorphism group A_n (and other groups related to A_n and S_n), and Dimitri Leemans, Maria Elisa Fernandes and Mark Mixer have extended some of this.

• Barry Monson and Egon Schulte (2009) used modular reduction techniques to construct new regular 4-polytopes of hyperbolic types $\{3, 5, 3\}$ and $\{5, 3, 5\}$ with a finite orthogonal group as automorphism group.

• Peter Brooksbank and Deborah Vicinsky (2010) showed that regular polytopes that have a 3-dimensional classical group as automorphism group must come from orthogonal groups.

• Ann Kiefer and Dimitri Leemans (2010) determined the regular polyhedra whose automorphism group is a Suzuki simple group $\mathrm{Sz}(q)$.

• Dimitri Leemans and Maria Elisa Fernandes (2011) proved that for every $n > 3$, the symmetric group S_n is the automorphism group of some regular r-polytope, for each r such that $3 \leq r \leq n-1$, and hence *for any given $r \geq 3$, all but finitely many S_n are the automorphism group of a regular r-polytope.*

Geometric and other considerations:

• Barry Monson and Egon Schulte wrote a series of five papers (2004–2009) on reflection groups and polytopes over finite fields, producing (for example) a catalogue of modular polytopes of small rank that are spherical or Euclidean.

• Peter McMullen (2004) classified abstract regular n-polytopes (and apeirotopes) that are faithfully realisable in a Euclidean space of dimension n (resp. $n-1$).

• Peter McMullen used similar techniques in order to classify 4-dimensional finite regular polyhedra (2007), and regular apeirotopes of dimension 4 (2009).

• Michael Hartley and Gordon Williams (2010) used methods for finding quotients of regular polytopes to obtain representations of the 14 sporadic Archimedean polyhedra.

• Isabel Hubard (2010) investigated 'two-orbit' polytopes, determining when the automorphism group is transitive on the faces of each rank, and used this to completely characterise the groups of two-orbit polyhedra.

• Mark Mixer (PhD) investigated the layer graphs (showing incidence between two layers) of regular polytopes, esp. the medial layer graph of regular n-polytopes for even n.

Properties of chiral polytopes:

• Asia Weiss and Isabel Hubard (2005) proved that every self-dual chiral polytope of odd rank admits a polarity, but that this is not true for even ranks.

• Asia Weiss, Egon Schulte and Isabel Hubard (2006) then showed how to construct chiral polyhedra from improperly self-dual chiral polytopes of rank 4, and regular polyhedra from properly self-dual ones.

Construction of chiral polytopes:

• Isabel Hubard, Marston Conder and Tomo Pisanski (2008) used computational group-theoretic methods to find subgroups of small index in Coxeter groups that are normal in the orientation-preserving subgroup but not in the group itself. This produced the smallest examples of finite chiral 3- and 4-polytopes, and also the first known finite chiral 5-polytopes, in both the self-dual and non-self-dual cases.

• Alice Devillers and Marston Conder (2009) found the first known finite chiral 6-, 7- and 8-polytopes, by group-theoretic construction for types $[3, 3, \ldots, 3, k]$.

• Daniel Pellicer (2010) devised a construction for chiral polytopes with prescribed regular facets (in some cases), and used this to prove the *existence of chiral d-polytopes, for all $d \geq 3$.*

The smallest regular polytopes in all ranks:

• Marston Conder answered a question by Daniel Pellicer (2010), by showing the following: For all $n \geq 9$, the regular n-polytopes with the smallest number of flags and no '2' in their type are polytopes of type $\{4, \overset{n-1}{\ldots}, 4\}$ with $2 \cdot 4^{n-1}$ flags. For $2 \leq n \leq 8$, the smallest regular n-polytopes have 6 flags (type $\{3\}$), 24 flags (type $\{3, 4\}$), 96 flags (type $\{4, 3, 4\}$), 432 flags (type $\{3, 6, 3, 4\}$), 1728 flags (type $\{4, 3, 6, 3, 4\}$), 7776 flags (type $\{3, 6, 3, 6, 3, 4\}$), and 31104 flags (type $\{4, 3, 6, 3, 6, 3, 4\}$). For $n \geq 9$, the above regular n-polytopes of type $\{4, \overset{n-1}{\ldots}, 4\}$ also have the smallest number of elements, and the smallest number of direct incidences (links in the Hasse diagram).

References:

Recent papers by the above-named contributors.