Lecture 3: Computational and group-theoretic methods

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In the study of maps and polytopes (and other discrete structures) with a high degree of symmetry, frequently the most symmetric examples fall into classes with the property that the automorphism group of every member of the class is a quotient of some 'universal' group — e.g. automorphism groups of regular maps are quotients of triangle groups.

There are several ideas and techniques from combinatorial and computational group theory that can be very helpful in dealing with such families. Some of these are as follows:

- Schreier coset graphs
- Coset enumeration
- Subgroups of small index in finitely-presented groups
- Double-coset graphs

Schreier coset graphs

Let G be a group generated by a finite set $X = \{x_1, x_2, \ldots, x_d\}$.

Given any transitive permutation representation of G on a finite set Ω, we may form a graph with vertex-set Ω, and with edges of the form $\alpha — \alpha x_i$ for $1 \le i \le d$.

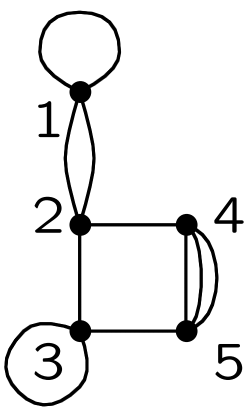
Similarly, if H is a subgroup of finite index in G, we may form a graph whose vertices are the right cosets of H and whose edges are of the form $Hg — Hgx_i$ for $1 \le i \le d$.

These two graphs are the same when Ω is the coset space $(G:H)$, or when H is the stabilizer of a point of Ω. It is called the Schreier coset graph $\Sigma(G, X, H)$.

Schreier coset graphs (cont.)

The Schreier coset graph $\Sigma(G, X, H)$ gives a diagrammatic representation of the natural action of G on cosets of H.

This can also be given by a coset table, e.g. as follows:



	x	y	x^{-1}	y^{-1}
1	2	1	2	1
2	1	3	1	4
3	3	5	3	2
4	5	2	5	5
5	4	4	4	3

when $x \mapsto (1,2)(4,5)$ and $y \mapsto (2,3,5,4)$

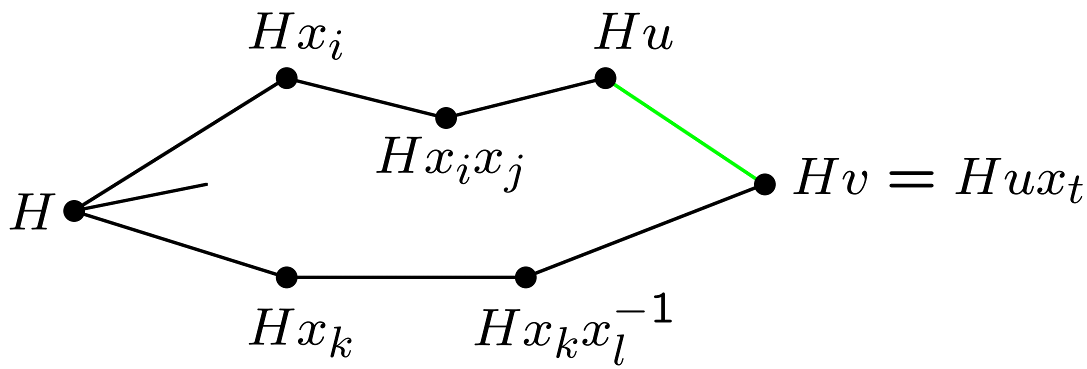
Some observations

- Generators of H correspond to circuits in the coset graph Σ based at the vertex labelled H

Why? Any path in Σ corresponds to a word $w = w(X)$ in the generators of G, and such a path from H is closed whenever $Hw = H$, which occurs if and only if $w \in H$.

- A Schreier transversal T for H in G corresponds to a spanning tree for the coset graph Σ

Why? Any path in a spanning tree based at the vertex H corresponds to a word $w = w(X)$, with initial sub-words corresponding to initial sub-paths of the given path.



- A Schreier generating-set for H in G corresponds to edges of the coset graph not used in the spanning tree

Why? See the illustration above, where solid lines represent edges of the spanning tree, and the green edge joins the vertex Hu to the vertex $Hv = Hux_t$, where u and v lie in the Schreier transversal T. This green edge completes a circuit corresponding to the Schreier generator

$$ux_t\overline{ux_t}^{-1} = ux_tv^{-1} \in H.$$

Application: Reidemeister-Schreier process

Given a finitely-presented group $G = \langle X \mid R \rangle$, and a subgroup H of finite index in G, we wish to find a presentation for H (in terms of generators and relations)

1) Construct the coset graph — using the coset table

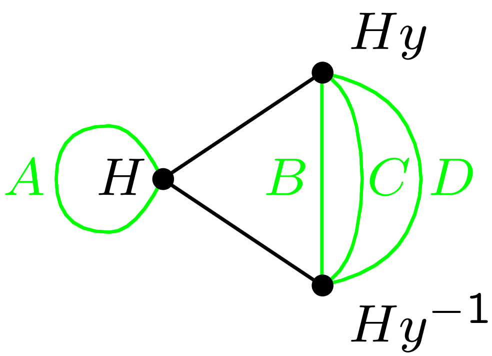
2) Take a spanning tree in the coset graph — this gives a Schreier transversal

3) Label the unused edges with Schreier generators

4) Apply each of the relators from R to each of the vertices in turn, to obtain the defining relations for H.

Example

Let $G = \langle\, x, y \,|\, x^2, y^3 \,\rangle$, and let H be the stabilizer of 1 in the permutation representation $x \mapsto (2,3)$, $y \mapsto (1,2,3)$:



Schreier generators

$A = x$

$B = y^3$ (= 1)

$C = yxy$

$D = y^{-1}xy^{-1}$

Relation $x^2 = 1$ gives new relations $A^2 = 1$ and $CD = 1$

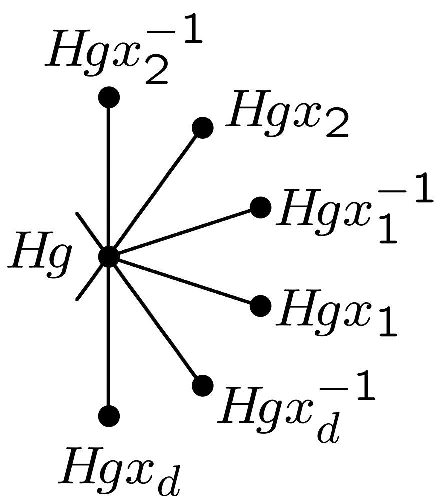
Relation $y^3 = 1$ gives new relation $B = 1$

Thus H has presentation $\langle A, C \,|\, A^2 \,\rangle$ via $A = x$ and $C = yxy$.

Another application: Ree-Singerman theorem

Let G be any group generated by permutations $x_1, x_2, \ldots, x_d$ on a set Ω of size n, such that $x_1 x_2 \ldots x_d = 1$ (identity), and let c_i be the number of orbits of $\langle x_i \rangle$ on Ω. Then for G to be transitive on Ω, one requires $c_1 + c_2 + \cdots + c_d \leq (d-2)n + 2$.

Proof. Embed the associated coset graph in an orientable surface such that the edges at each vertex are ordered thus:



Then $|V| = n$ and $|E| = dn$

while $|F| = \sum_{i=1}^{d} c_i + n$

and therefore

$$2 \geq |V| - |E| + |F| = \sum_{i=1}^{d} c_i + (2-d)n.$$

Coset diagrams — simplified coset graphs

To make a Schreier coset graph easier to work with, we can simplify it by

- deleting loops (that occur for fixed points of generators)
- using single edges for 2-cycles of involutory generators
- ignoring the effect of redundant generators.

Coset graphs for actions of $(2, k, m)$ triangle groups

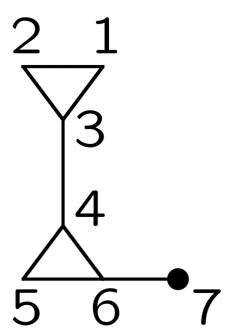
$$\langle\, x, y, z \mid x^2 = y^k = z^m = xyz = 1 \,\rangle$$

can be simplified even further, by using heavy dots only for fixed points of y, and polygons for non-trivial cycles of y.

The resulting figures are called (Schreier) coset diagrams, rather than coset graphs.

Example

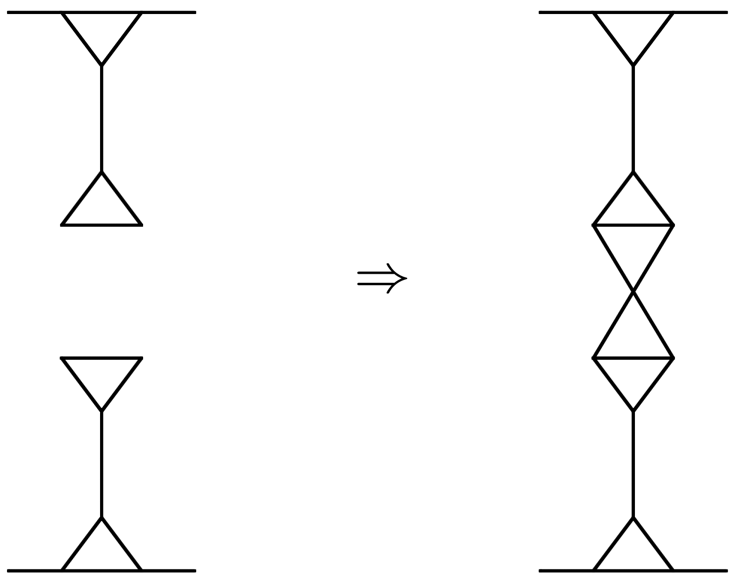
Below is a coset diagram for an action of the $(2,3,7)$ triangle group $\langle\, x, y, z \mid x^2 = y^3 = z^7 = xyz = 1 \,\rangle$ on 7 points:



$$x \mapsto (3,4)(6,7)$$
$$y \mapsto (1,2,3)(4,5,6)$$
$$z \mapsto (1,4,7,6,5,3,2)$$

Composition of coset diagrams

Often two coset diagrams for the same group G on (say) m and n points can be composed to produce a transitive permutation representation of larger degree $m + n$, e.g.



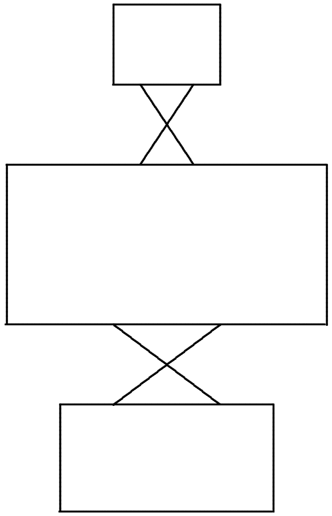
What effect does this have?

Strange things can happen! For example, consider coset diagrams for transitive actions of the $(2, 3, 7)$ triangle group. One can join together three such diagrams D1, D2, D3:

D1 on 14 points, where the permutations generate a group isomorphic to $\mathrm{PSL}(2, 13)$

D2 on 64 points, where the permutations generate a group isomorphic to A_{64}

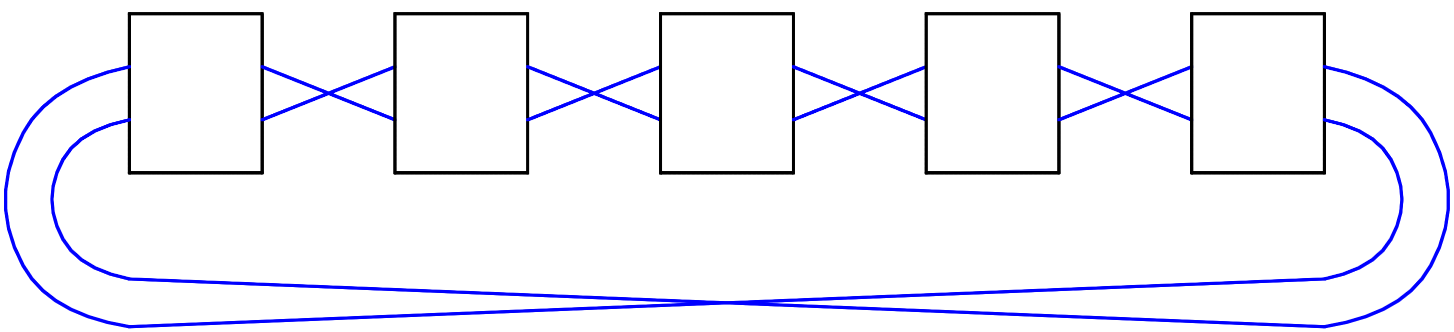
D3 on 22 points, where the permutations generate a group isomorphic to A_{22}



to get a diagram on $14{+}64{+}22 = 100$ points, where the permutations generate the Hall-Janko group of order 604800.

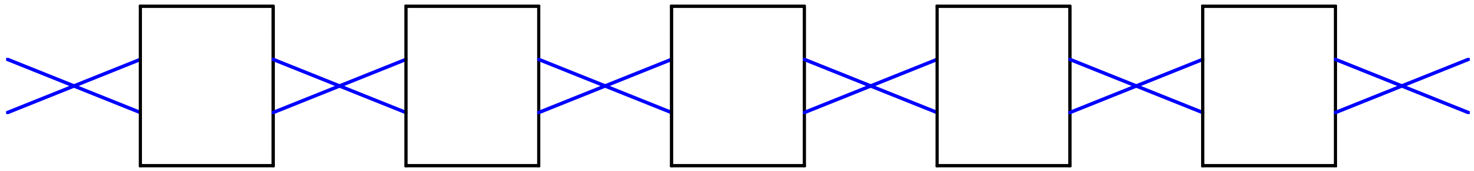
Abelian covers

In some cases, where a coset diagram for a group H may be joined together to another copy of itself in two different places, it is possible to string together n copies of the diagram into a circular chain (like a necklace) and get a new diagram in which the permutations generate a larger group G with an abelian normal subgroup K of exponent n such that $G/K \cong H$.



Proving groups are infinite

When that is possible, string together an infinite number of copies of the diagram:



and get an infinite group!

This method can be used to prove that certain finitely-presented groups are infinite. It is equivalent to showing that some subgroup of finite index has infinite abelianization ... also achievable by the Reidemeister-Schreier process.

Alternating and symmetric quotients

- If Diagrams P and Q each have two 'handles' (for attaching to other diagrams), and have m points and n points respectively, then we can string together p copies of P and q copies of Q and get a large diagram on $m = ap + bq$ points

- In particular, if $\gcd(p, q) = 1$, then $m = ap + bq$ can be any sufficiently large positive integer

- We can sometimes adjoin a single copy of an extra diagram R (with r points) to 'break symmetry', and make the permutations induced on the large diagram generate the alternating group A_{m+r} or the symmetric group S_{m+r}.

- In this way, we can sometimes obtain all but finitely many A_n or S_n as quotients of a given finitely-presented group.

Chiral quotients

- Consider coset diagrams for a given triangle group

- If a (large) diagram P has mirror symmetry and another (small) diagram Q has no mirror symmetry, then composing a copy of P and a copy of Q can produce a larger diagram that has no mirror symmetry — this is another form of 'symmetry breaking'

- In this way, we can sometimes construct infinitely many chiral maps of a given hyperbolic type.

Some applications

- Every $(2,3,m)$ triangle group has all but finitely many alternating groups A_n among its homomorphic images

- There are infinitely many chiral maps of type $\{3,k\}$, for each $k \geq 7$

- Every Fuchsian group has all but finitely many alternating groups A_n among its homomorphic images [Brent Everitt]

- There are infinitely many 5-arc-transitive connected finite 3-valent graphs, and infinitely many 7-arc-transitive connected finite 4-valent graphs

- There are infinitely many 5-arc-transitive Cayley graphs of valency 3, and infinitely many 7-arc-transitive Cayley graphs of valency $3^t + 1$ for each $t \geq 1$.

Summary

Schreier coset graphs

- depict transitive actions of groups
- illustrate/assist the Reidemeister-Schreier process
- help prove the Ree-Singerman theorem
- may help build large quotients of a group from small ones
- can be used to prove certain groups are inifinite

Next:

How do we find good examples in the first place?

Coset enumeration

Let $G = \langle\, X \,|\, R \,\rangle$ be any finitely-presented group, and let H be the subgroup generated by a given finite set Y of words on the alphabet $X = \{x_1, \ldots, x_m\}$.

Methods exist for systematically enumerating the cosets Hg for $g \in G$, by using the generators and relations to help construct the coset table:

	x_1	x_2	$\ldots$	x_1^{-1}	x_2^{-1}	$\ldots$
1	2	3		4		
2				1		
3					1	
4	1					
$\vdots$						

Each relation from the defining presentation $\langle X \mid R \rangle$ for G forces pairs of cosets to be equal: $Hg\,r = Hg$ for all $g \in G$.

The same thing happens on application of each generator $y \in Y$ to the trivial coset H: $Hy = H$.

New cosets are defined (if needed), and all such coincidences are processed, until the coset table either 'closes' or has too many rows.

If the coset table closes with n cosets, then $|G : H| = n$. Moreover, the coset table gives us the natural permutation representation of G on the right coset space $(G\!:\!H)$.

If it does not close, then the index $|G\!:\!H|$ could be infinite, or just too large to be found (or it might even be small but the computation was not given enough resources).

Low index subgroup methods

On the previous slide, we had a finitely-presented group $G = \langle X \mid R \rangle$, and looked at enumerating cosets of a given subgroup H (of finite index in G). The basic method was established by Todd and Coxeter (1936), and further developed by other such as Leech and Havas.

But: how do we find candidates for H?

Answer: For prescribed positive integer n, we can find all subgroups of index up to n in G (up to conjugacy) by a systematic enumeration of coset tables with at most n rows.

The basic method is due to Charles Sims (1970s).

Forcing coincidences

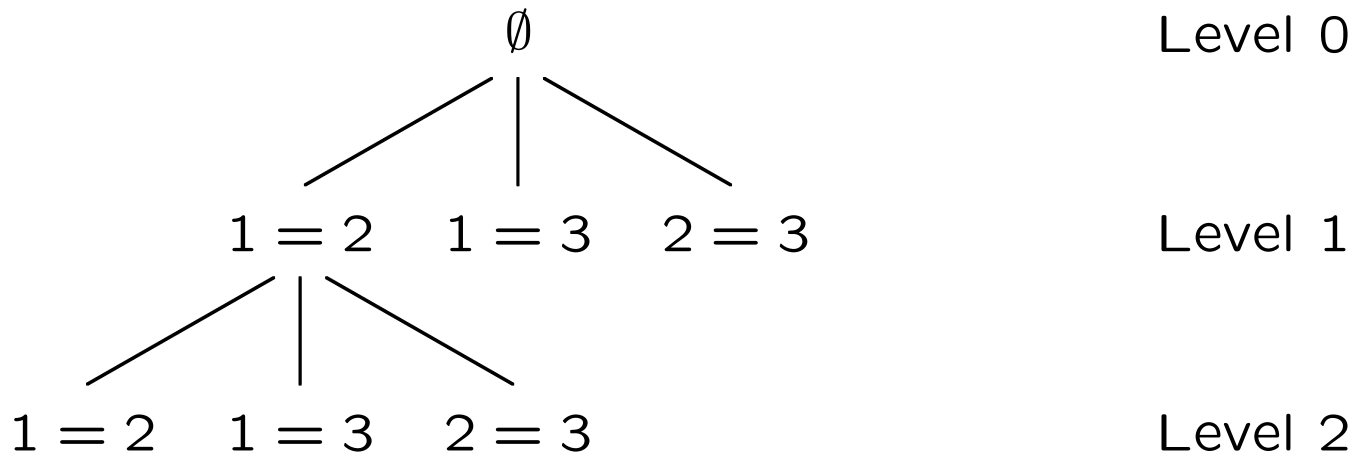
The key to the standard 'Low index subgroups algorithm' is to define more than n cosets, and then force coincidences between them, using the fact that $Ha = Hb \Leftrightarrow ab^{-1} \in H$.

The algorithm starts with the identity subgroup and attempts to enumerate its right cosets, constructing a partial transversal $\{u_1, u_2, u_3, \dots\}$. Then (or at any stage) if more than n cosets are defined, all possible concidences between two cosets Hu_i and Hu_j are considered, for $1 \le i < j \le n+1$.

Often such a coincidence will be found to lead to a subgroup H that is conjugate to one found previously, in which case that coincidence is rejected and the next one is looked at. If not rejected, then $u_i u_j^{-1}$ is added to a (partial) set of generators for H, and the search continues.

Branching/backtrack process

Systematic enumeration of coincidences between cosets (and adding new generators for H) sets up a branching process:



A backtrack search will terminate (given sufficient time and memory), by Schreier's theorem: every subgroup of finite index in a finitely-generated group is itself finitely-generated.

Example:

Let $G = \langle\, x, y \,|\, x^2 = y^3 = 1 \,\rangle$, which is the modular group, and look for subgroups of index up to 4.

#	Coincidences	Index	Generators
1	$1 = 2,\ 1 = 2$	1	$x,\ y$
2	$1 = 2,\ 2 = 4,\ 3 = 4$	3	$x,\ yxy^{-1},\ y^{-1}xy$
3	$1 = 2,\ 2 = 4,\ 4 = 5$	4	$x,\ yxy^{-1},\ y^{-1}xy^{-1}xy$
4	$1 = 2,\ 3 = 4$	3	$x,\ y^{-1}xy^{-1}$
5	$1 = 3,\ 2 = 3$	2	$y^{-1},\ xy^{-1}x$
6	$1 = 3,\ 4 = 5$	4	$y^{-1},\ xy^{-1}xy^{-1}x$

Low index normal subgroups

Small homomorphic images of a finitely-presented group G can be found as the groups of permutations induced by G on cosets of subgroups of small index. This gives G/K where K is the core of H, but produces only images that have small degree faithful permutation representations.

Alternatively, the (standard) low index subgroups method can be adapted to produce only normal subgroups.

A new method was developed recently by Derek Holt and his student David Firth, which systematically enumerates all possibilities for a composition series of a factor group G/K, where K is a normal subgroup of small index in G.

This method has produced lots of symmetric structures.

Summary

- Coset enumeration

- Standard ‘Low Index Subgroups’ algorithm

- New variant for finding normal subgroups only

- These two methods can help find all small degree transitive permutation representations and all small quotients of a given finitely-presented group.

(Sabidussi) Double-coset graphs

Let G be a group, H a subgroup of G, and a an element of G such that $a^2 \in H$. Now define a graph $\Gamma = \Gamma(G, H, a)$ by

$$V(\Gamma) = \{Hg : g \in G\}$$

$$E(\Gamma) = \{\{Hx, Hy\} : x, y \in G \mid xy^{-1} \in HaH\}.$$

Then G induces a group of automorphisms of Γ by right multiplication, since $(xg)(yg)^{-1} = xgg^{-1}y^{-1} = xy^{-1} \in HaH$ whenever $\{Hx, Hy\} \in E(\Gamma)$).

This action is vertex-transitive (since $(Hx)x^{-1}y = Hy$), with vertex-stabilizer $G_H = \{g \in G : Hg = H\} = H$ itself, which acts transitively on the neighbours Hah (for $h \in H$) of H.

Thus $\Gamma = \Gamma(G, H, a)$ is arc-transitive!

Example: a double-coset graph for A_5

Let $G = A_5$ (the alternating group on 5 points), and take $H = \langle\,(1,2,3),(1,2)(4,5)\,\rangle \cong S_3$ and $a = (1,4)(2,5)$.

Then the coset graph $\Gamma(G,H,a)$ is a connected arc-transitive 3-valent graph of order 10 which turns out to be the Petersen graph.

This can also be constructed as a double-coset graph for $G = S_5$, using $H = \langle\,(1,2,3),(1,2),(4,5)\,\rangle \cong S_3 \times C_2$ and element $a = (1,4)(2,5)$.

In fact S_5 is the full automorphism group.

Special case: 3-valent symmetric graphs

• It is known that if Γ is a finite connected 3-valent symmetric graph, then $G = \mathrm{Aut}\,\Gamma$ is a quotient of one of the seven finitely-presented groups G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2, G_5 described earlier

• Conversely, if G is any non-degenerate quotient of one of those groups, then we can use the double-coset graph construction to obtain a finite connected 3-valent symmetric graph Γ on which G acts as a group of automorphisms

• For example, $G = A_5$ is a quotient of the group

$$G_2^1 = \langle\, h, a, p \,|\, h^3 = a^2 = p^2 = 1,\ apa = p,\ php = h^{-1} \,\rangle$$

via $h \mapsto (1,2,3)$, $a \mapsto (1,4)(2,5)$, $p \mapsto (1,2)(4,5)$, and from this we get the Petersen graph.

Small 3-valent symmetric graphs

- Take each one of G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2, G_5 in turn

- Use the 'Low Index Normal Subgroups' algorithm (due to Firth & Holt) to find all normal subgroups of index up to n

- For each normal subgroup K, let G be the quotient of the given group by K, and use the double-coset graph construction to obtain a finite connected 3-valent symmetric graph Γ on which G acts as a group of automorphisms

- Check whether G is the full automorphism group of Γ (using GAP or Magma); discard the graph if it's not

- This approach has provided a census of all such graphs on up to 10,000 vertices [MC, 2011].