6/29/11

The Approximability of Constraint Satisfaction Problems

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$\boxed{\text{See Notes on Web}}$

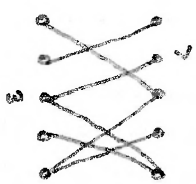
Inapproximability

PCP Thm $\Leftrightarrow$ $(\alpha, 1)$-distinguishing 3SAT is NP-hard (for some $\alpha < 1$).

APX-hardness (α, β)-distinguishing is hard $(\alpha < \beta)$
APX-hardness for lots of problems.

Label-Cover (R) binary CSP

3SAT



$$\underset{w \to v}{\pi} : \underset{\{1,2,\ldots,6\}}{[6]} \to [6]$$

$$\underset{w \to v}{\pi}(\ell(w)) = \ell(v)$$

(instance graph
primal graph)

alphabet $\{1, 2, \ldots, 6\}$

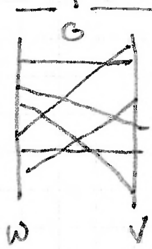
PCP Thm $\Rightarrow \exists\, \alpha < 1$ s.t. $(\alpha, 1)$-distinguishing $LC(6)$ is NP-hard

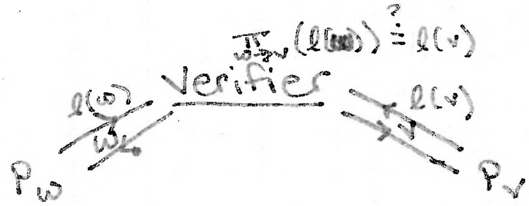
Parallel Repetition Thm| (Raz, '95) $\forall\, \varepsilon\ \exists\ R = R(\varepsilon)$ s.t. $(\varepsilon, 1)$-distinguishing $LC(R)$ is NP-hard (on bipartite primal graph).

- $R \geq \frac{1}{\varepsilon}$
- $R \leq \left(\frac{1}{\varepsilon}\right)^c$ for absolute constant c.

Thm| (Håstad) $\forall\, \gamma > 0$ $\left(\frac{7}{8} + \gamma, 1\right)$-distinguishing 3SAT is NP-hard.

2-prover game





$\ell : V \cup W \to \{1, \ldots, R\}$

Exercise| Value of game = Opt(G)

notes by Megan Heenehan

<u>Goal</u>: Check $\pi_{w\to v}(\ell(w)) = \ell(v)$ without reading $\ell(w)$ and $\ell(v)$ entirely, but rather only 3 bits (and checking 3SAT constraint on them).

<u>Idea</u>: Provers <u>hold</u> not $\ell(w)$ and $\ell(v)$, but some encoding of them as per some code.

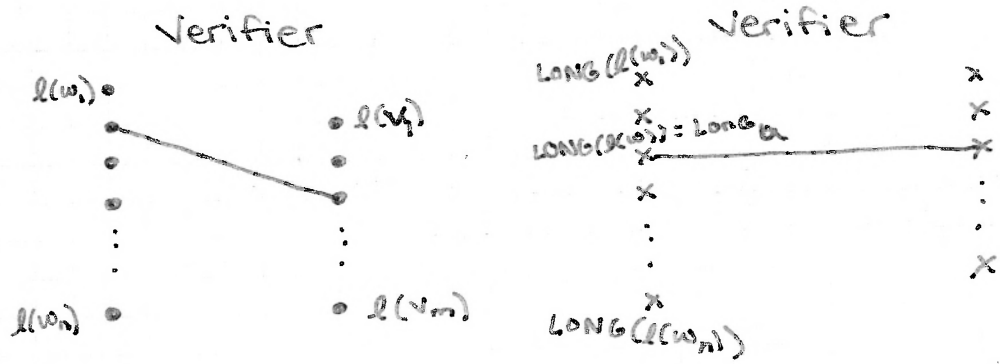
(Bellare, Goldreich, Sudan '96) <u>Long Code</u>

$a \in [R]$ $\quad \mathrm{LONG}_a \in \{0,1\}^{2^R}$
a (log R bits)

$$\mathrm{LONG}_a : \{0,1\}^R \to \{0,1\}$$
$$\forall x \in \{0,1\}^R$$
$$\mathrm{LONG}_a(x) = x_a \text{ (projection on coordinate a)}$$

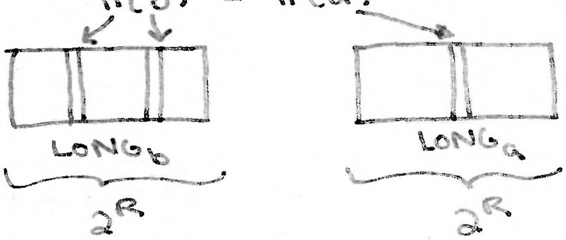
$f : [R] \to \{0,1\}$

Dictator Function

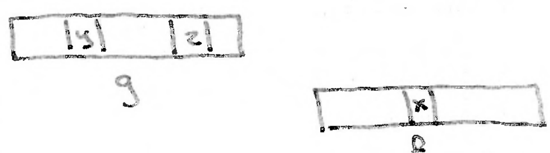


$\pi_{w\to v} = \pi$

$\pi(b) \stackrel{?}{=} \pi(a)$



Check $f(x) \vee g(y) \vee g(z)$



$(x, y, z) \in D$ on $(\{0,1\}^R)^3$

$\forall j \quad y_j \vee z_j \vee x_{\pi(j)} = 1$
$(j \in \{1, \dots, R\})$

"Folding"

$$f(x) \to f(x) \vee g(y) \vee g(z)$$
$$f(x) \to f(\bar{x}) \vee g(y) \vee g(z)$$

f – LONG_a

$f(\bar{x}) = \bar{x}_a$

3-LIN $\quad \left\{ \begin{matrix} x \oplus y \oplus z = 1 \\ x \oplus y \oplus z = 0 \end{matrix} \right\}$

Thm $\forall\ \gamma, \varepsilon > 0$, $\left(\frac{1}{2} + \gamma, 1 - \varepsilon\right)$-distinguishing 3-LIN is NP-hard.

reduction

$x \oplus y \oplus z = 0 \to (\bar{x} \vee \bar{y} \vee \bar{z})(\bar{x} \vee y \vee z)(x \vee \bar{y} \vee z)(x \vee y \vee \bar{z})$

Cor $\forall\ \gamma, \varepsilon > 0$ $\left(\frac{7}{8} + \gamma, 1 - \varepsilon\right)$-distinguishing 3SAT is NP-hard.

(Approximation Resistance connection)

[redacted] Reduce $(\delta, 1-\varepsilon)$-distinguishing Unique Games (R) $(UG(R))$ to $\left(\frac{1}{2} + \gamma, 1 - \varepsilon\right)$-disting. 3LIN.

Dictatorship Testing

Is $g = \text{LONG}_a$ for some a?

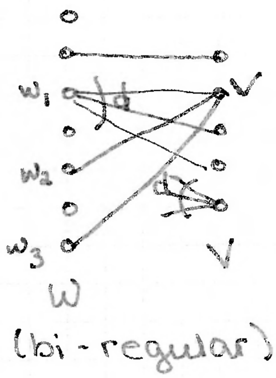
$g: \{0,1\}^R \to \{0,1\}$

If g is a Long Code
$g(x) \oplus g(y) = g(x \oplus y)$

$x_1 \oplus x_2 \oplus \ldots \oplus x_R$

μ · ε-biased
$\mu_i = 0$ with prob. $1 - \varepsilon$
$\quad 1$ with prob ε
indep. for all i

Verifier



① Pick $v \in V$ at random

② Pick $w_1, w_2, w_3 \in N(v)$ at random (sample with replacement)

③ Pick (x, y, μ) as in that test.

$z = x \oplus y \oplus \mu$

④ $x' = x \circ \pi_{w_1 \to v}$ $\quad (x \circ \pi)_j \quad x_{\pi(j)}$

$y' = y \circ \pi_{w_2 \to v}$

$z' = z \circ \pi_{w_3 \to v}$

⑤ With prob. $1/2$ check that

$f_{w_1}(x') \oplus f_{w_2}(y') \oplus f_{w_3}(z') = 0$

and with prob. $1/2$ check

$f_{w_1}(x') \oplus f_{w_2}(y') \oplus f_{w_3}(\bar{z}) = 1$

Completeness

UG instance is $(1-\varepsilon)$-satisfiable

$\Rightarrow$ 3LIN instance is $(1-4\varepsilon)$-satisfiable.

$f_w = \mathrm{LONG}_{\ell(w)}$ | $\ell : V \cup W \to R$ is a labeling satisfying $(1-\varepsilon)$ of the UG constraints.

(v, w_1) is a random edge

$f_{w_1}(x') = x'_{\ell(w_1)} \underset{\uparrow}{=} x_{\pi_{w_1 \to v}(\ell(w_1))} = x_{\ell(v)}$

with prob. $1-\varepsilon$

With prob. $1 - 3\varepsilon$

$x_{\ell(v)} \oplus y_{\ell(v)} \oplus z_{\ell(v)} \stackrel{?}{=} 0 \iff \mu_{\ell(v)} \stackrel{?}{=} 0$

Thm

then

$\gamma \equiv$

Hugel

Pr [

Thm/ If the UG instance is at most δ-satisfiable then verifier accepts with prob. $\leq 1/2 + \gamma$ where $\gamma \equiv O\left(\left(\frac{\delta}{\varepsilon}\right)^{1/5}\right)$.

Hugely convenient notational switch:

$$f_w : \{0,1\}^R \longrightarrow \{1,-1\}$$
$$b \longmapsto (-1)^b$$

$$f(x') \oplus f(y') \oplus f(z') = 0$$

$$\Pr[\text{verifier accepts}] = \rho = \frac{1}{2} \mathop{\mathbb{E}}_{\substack{v, w_1, w_2, \\ w_3, x, y}} \left[\frac{1 + f_{w_1}(x') f_{w_2}(y') f_{w_3}(z')}{2} \right] + \frac{1}{2} \mathbb{E} \left[\frac{1 - f_{w_1}(x') f_{w_2}(y') f_{w_3}(z')}{2} \right]$$