6/29/11

Universal Algebra for CSP   Ross Willard

$\mathbb{K}_3 = (A, \neq_A) \quad A = \{0,1,2\}$

$\mathbb{G}_1 = (2; \{R_{abc} : abc \in \{0,1\}^3\}) \qquad R_{abc} = 2^3 \setminus \{(a,b,c)\}$
(gives 3 Sat)

$\mathbb{G}_1$ is pp-constructible in $\mathbb{K}_3$ ($\mathbb{G}_1 \leq_{ppc} \mathbb{K}_3$)

Pf Sketch) Will define (on $\mathbb{K}_3$)
$F \subseteq A^9$
$E \subseteq A^{18}$ (an equiv. rel. on F)
For abc $S_{abc} \subseteq A^{27}$ ($S_{abc} \subseteq F^3$)

F will have exactly two E-blocks O, I

~~e.g. $S_{010} = \{(x_1, y_1, z_1, x_2, y_2, z_2$~~

e.g. $S_{010} = \{(x_1, \dots, x_9, y_1, \dots, y_9, z_1, \dots, z_9) \in A^{27} :$
$\bar{x} \in F$ & $\bar{y} \in F$ & $\bar{z} \in F$ and $\neg(\bar{x} \in O$ & $\bar{y} \in I$ & $\bar{z} \in O)\}$

Then $(F/E; \{S_{abc}/E : abc \in 2^3\}) \cong \mathbb{G}_1$
$\quad\ \ \overset{\shortparallel}{\{O, I\}}$

Finally, show each of F, E, S_{abc} are pp-definable in $\mathbb{K}_3$.

A^9 "=" $A^{3\times 3}$

$$O = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{bmatrix}, \dots \right\}$$

$$I = \left\{ \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}, \dots \right\}$$

$F = O \cup I$.

$E = O \times O \cup I \times I$

S_{abc} defined as explained earlier.

Show F, E, S_{abc} pp-def in $\mathbb{K}_3$.

F is the free algebra on 2 generators in HSP(Pol Alg $\mathbb{K}_3$)
Elements of F are functions $A^2 \to A$.

notes by Megan Heenehan

$F = \{$2-ary polymorphisms of $\mathbb{K}_3\}$

are homomorphisms $\mathbb{K}_3^2 \to \mathbb{K}_3$

Defn of F : $\left[x_{ij}\right]_{3\times 3} \in F$ iff

$$\bigwedge_{i \neq i' \; j \neq j'} x_{ij} \neq x_{i'j'} \quad \text{(a pp-formula of } \mathbb{K}_3\text{)}$$

Show E is pp-definable in $\mathbb{K}_3$.
First show E is compatible with $\mathrm{Pol}(\mathbb{K}_3)$
Pick $\sigma \in \mathrm{Pol}(\mathbb{K}_3)$ WLOG arity$(\sigma) = 1$ and σ is a permutation of A.
Suppose

$$\left(\begin{bmatrix} x_{11} & x_{12} & \cdots \\ \vdots & & \\ & & \end{bmatrix}, \begin{bmatrix} y_{11} & y_{12} & \cdots \\ \vdots & & \\ & & \end{bmatrix} \right) \in E$$

Apply σ : $\left(\begin{bmatrix} \sigma(x_{11}) & \sigma(x_{12}) & \cdots \\ & \vdots & \\ & & \end{bmatrix}, \begin{bmatrix} \sigma(y_{11}) & \sigma(y_{12}) & \cdots \\ & \vdots & \\ & & \end{bmatrix} \right) \overset{?}{\in} E$

Obviously true since O, I both stable under σ.
By the Thm (Bod/Geiger) E is pp-definable in $\mathbb{K}_3$.

Similar argument works for S_{abc}.

If you want explicit formulas
Recipe: $|E| = 72$
Guarantees a pp-formula defining E with 3^{72} variables

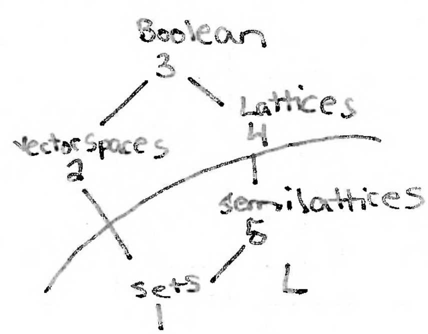
$|S_{abc}| = 7 \cdot 6^3$ $3^{7 \cdot 6^3}$ variables in recipe-formula for S_{abc}.

<u>Tame Congruence Theory</u> (TCT)
Hobby, McKenzie

For finite algebras

$\underset{\sim}{A} \longmapsto$ set of types

$\emptyset \neq \mathrm{type}(\underset{\sim}{A}) \subseteq \{1, \ldots, 5\}$



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To each downset L in poset on previous page get a <u>dichotomy</u>:

$\forall$ finite $\underset{\sim}{A}$, either

HSP($\underset{\sim}{A}$) admits some type from L
OR
HSP($\underset{\sim}{A}$) omits all types from L

HSP($\underset{\sim}{A}$) admits some type from L	HSP($\underset{\sim}{A}$) omits all types from L
$\equiv$ $\exists$ some structure of some kind compatible with a ~~[illegible]~~ member of HSP($\underset{\sim}{A}$)	$\equiv$ $\exists$ some identities which can be satisfied in the clone of $\underset{\sim}{A}$. "A Maltsev condition"

These dichotomies from TCT are surprisingly robust
- relevant to complexity of CSPs
- Gave us the Alg. Dichot. Conj.

Focus on the dichotomy for "omitting type 1"

Let $\mathbb{G}$ be any finite idempotent relational structure, $\underset{\sim}{A}$ = Pol Alg($\mathbb{G}$) (an idempotent algebra).

HSP($\underset{\sim}{A}$) admits 1	HSP($\underset{\sim}{A}$) omits 1
(0) HS($\underset{\sim}{A}$) contains $(2;\emptyset)$ $\downarrow$ (1) HSP($\underset{\sim}{A}$) contains $(2;\emptyset)$ $\Vert$ (2) $\mathbb{G}_1 \leq_{ppc} \mathbb{G}$ easy [illegible] example (3) $\mathbb{K}_3 \leq_{ppc} \mathbb{G}$ $\downarrow$ (4) $\exists$ graph $\mathbb{T}$ with ~~[illegible]~~ core($\mathbb{T}$) $= \mathbb{K}_3$ $\mathbb{T} \leq_{ppc} \mathbb{G}$ "$\mathbb{K}_3$-partitionability" $\downarrow$ (5) $\exists$ graph $\mathbb{T}$ containing $\mathbb{K}_3$ with $\mathbb{T}_{[illegible]} \leq_{ppc} \mathbb{G}$ $\downarrow$	(a) $\underset{\sim}{A}$ satisfies Malstev condition not satisfiable in $(2;\emptyset)$ see book on TCT (b) $\underset{\sim}{A}$ has a <u>Taylor</u> operation (in some # of variables) no TCT (c) ($f(x_1,\dots,x_k)$ idempotent $f(x,\dots,x) \approx x$ $\forall\ 1 \leq i \leq k$ $\exists$ some identity so that $\forall x$ ~~[illegible]~~ $\forall y$ $f(\_\_, \underset{i}{x}, \_\_) \approx f(\_\_, \underset{i}{y}, \_\_)$ variables <u>Ex</u> maltsev op $f(x,x,y) \approx y = f(y,y,y)$ $\quad$ $f(x,y,y) \approx x = f(x,x,x)$

- (b) $\exists$ non-bipartite graph $\mathbb{T}$, $\mathbb{T} \leq_{ppc} \mathbb{G}$

Bulatov's proof
Hell - Nesetril Thm

(c) $\exists$ weak NU (near unanimity) operation (of some arity)

NU: $f(x,\dots,x,y) = x$
$f(y,x,\dots,x) = x$
$f(x,y,x,\dots,x) = x$

weak NU (WNU):
$f(x,x,\dots,x) \approx x$
$f(x,x,\dots,x,y) \approx f(y,x,\dots,x) \approx f(x,y,x,\dots,x)$

(d) $\exists$ cyclic op (of some arity)
$f(x,\dots,x) \approx x$
$f(x_1,x_2,\dots,x_k) \approx f(x_2,\dots,x_k,x_1) \approx \dots$

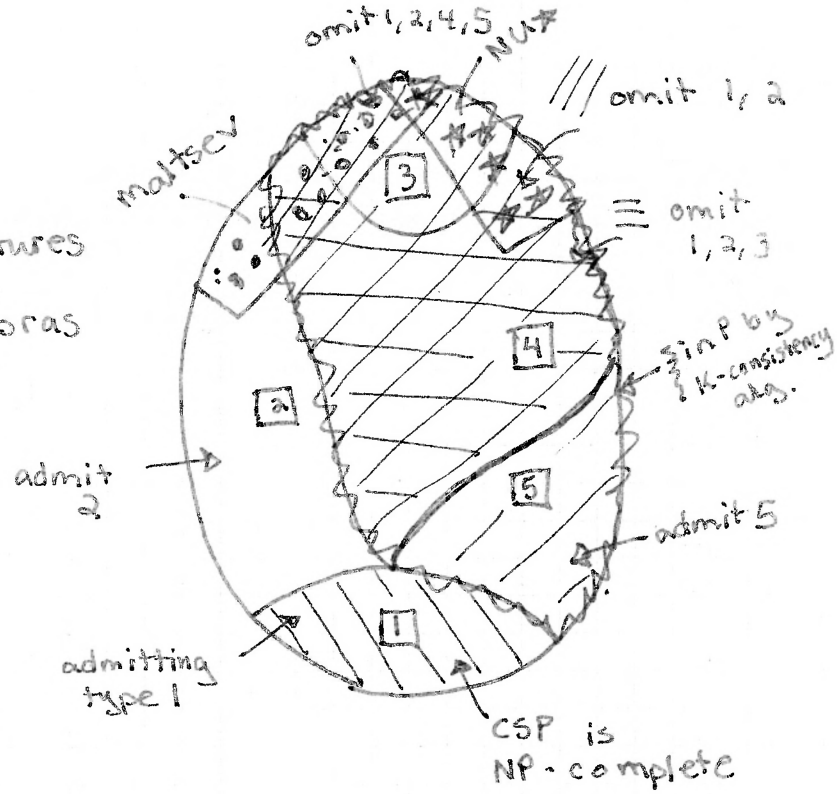
(e) $\forall$ prime $p > |A|$ $\exists$ cyclic op. of arity p.

(f) $\exists$ Sigers operation:
6-ary idempotent op
$f(x,y,z,x,y,z) \approx f(y,z,x,z,x,y)$

(b)

Exercise | Show (f) $\equiv \neg(5)$ (Hint: Like Maltsev's Thm)

All
finite idempotent structures
$\equiv$
finite idempotent algebras



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