

TREE ON TOP OF MALTSEV

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ABSTRACT. Let \mathbf{B} be an idempotent algebra, $\beta \in \text{Con } \mathbf{B}$ such that \mathbf{B}/β is term equivalent to a semilattice whose order is a rooted tree, and each β -block is Maltsev. Then $\text{CSP}(\mathbf{B})$ is tractable.

Let $\mathbf{B} = (B; F)$ be an algebra and t be a unary idempotent polynomial of \mathbf{B} . The *retract* of \mathbf{B} via t is the algebra $t(\mathbf{B}) = (t(B); \{t \circ f : f \in F\})$. A template \mathcal{B} for the constraint satisfaction problem is a set of finite idempotent algebras of similar type closed under taking subalgebras, homomorphic images and retracts via idempotent unary polynomials, but containing only one algebra of each isomorphism type. An *instance*

$$\mathcal{A} = \{ \mathbf{B}_i \in \mathcal{B}, \mathbf{R}_I \leq \prod_{i \in I} \mathbf{B}_i \mid i \in V, I \in S \}$$

of the constraint satisfaction problem $\text{CSP}(\mathcal{B})$ consists of a set V of variables, a domain set $\mathbf{B}_i \in \mathcal{B}$ for each variable $i \in V$, a set $S \subseteq P(V)$ of constraint scopes, and a constraint relation $\mathbf{R}_I \leq \prod_{i \in I} \mathbf{B}_i$ for each scope $I \in S$. A *solution* of \mathcal{A} is a function $f \in \prod_{i \in V} B_i$ such that $f|_I \in R_I$ for each scope $I \in S$.

1. CONSISTENT MAPS

Definition 1. Let \mathcal{A} be an instance of $\text{CSP}(\mathcal{B})$. A set $p = \{p_i \mid i \in V\}$ of maps is *consistent* with \mathcal{A} if for all $i \in V$ the map p_i is a unary polynomial of \mathbf{B}_i , and for every scope $I \in S$ and tuple $r \in R_I$ the tuple $p|_I(r) = \langle p_i(r_i) : i \in I \rangle$ is also in R_I . We say that p is *permutational*, if each p_i is a permutation, and it is *idempotent*, if $p_i(p_i(x)) = p_i(x)$ for all $i \in V$.

Clearly, every consistent set $p = \{p_i : i \in V\}$ of maps can be iterated to obtain an idempotent one $p' = \{p_i^k : i \in V\}$ where $k = (\max_{i \in I} |B_i|)!$ for example.

Definition 2. Let \mathcal{A} be an instance of $\text{CSP}(\mathcal{B})$ and $p = \{p_i \mid i \in V\}$ be a consistent set of idempotent unary polynomials. The *retraction* of \mathcal{A} via p is the new instance $p(\mathcal{A})$ of $\text{CSP}(\mathcal{B})$ defined as

$$p(\mathcal{A}) = \{ p_i(\mathbf{B}_i), p|_I(\mathbf{R}_I) \mid i \in V, I \in S \}.$$

It easily follows from the definitions that the relation

$$p|_I(\mathbf{R}_I) = \{ p|_I(r) \mid r \in R_I \} = R_I \cap \prod_{i \in I} p_i(B_i)$$

is a subuniverse of $\prod_{i \in I} p_i(\mathbf{B}_i)$.

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Lemma 3. *Let \mathcal{A} be an instance of $\text{CSP}(\mathcal{B})$ and p be a consistent set of idempotent unary polynomials. Then \mathcal{A} has a solution if and only if $p(\mathcal{A})$ does.*

Proof. Since $p_i(B_i) \subseteq B_i$ and $p|_I(R_I) \subseteq R_I$, any solution of $p(\mathcal{A})$ is a solution of \mathcal{A} . Conversely, if f is a solution of \mathcal{A} , then the function $p \circ f = \langle p_i(f_i) : i \in V \rangle$ is a solution of $p(\mathcal{A})$. \square

Definition 4. Let \mathcal{A} be an instance of $\text{CSP}(\mathcal{B})$ and t be a binary term such that $t(x, t(x, y)) = t(x, y)$. For an element $b \in B_i$ we put $t_b(x) = t(b, x)$, which is an idempotent polynomial of \mathbf{B}_i . The *decomposition* of \mathcal{A} via t is the new instance $t(\mathcal{A})$ of $\text{CSP}(\mathcal{B})$ defined as

$$t(\mathcal{A}) = \{ \mathbf{B}_{i,b}, \mathbf{R}_{I,r}, \mathbf{T}_{i,B_i} \mid (i, b) \in V', (I, r) \in S', (i, B_i) \in U' \},$$

where

$$V' = \{ (i, b) \mid i \in V, b \in B_i \}$$

is the set of variables,

$$\mathbf{B}_{i,b} = t_b(\mathbf{B}_i) = \{ t(b, x) \mid x \in B_i \}$$

are the domains,

$$S' = \{ (I, r) \mid I \in S, r \in R_I \}$$

$$U' = \{ (i, B_i) \mid i \in V \}$$

are sets of scopes where

$$(I, r) = \{ (i, r_i) \mid i \in I \} \quad \text{and} \quad (i, B_i) = \{ (i, b) \mid b \in B_i \},$$

and

$$\mathbf{R}_{I,r} = t_r(\mathbf{R}_I) = \{ \langle t(r_i, s_i) : i \in I \rangle \mid s \in R_I \},$$

$$\mathbf{T}_{i,B_i} = \text{Sg}_{\mathbf{B}_i^*} \{ \langle t(b, c) : b \in B_i \rangle \mid c \in B_i \}$$

are the relations where $\mathbf{B}_i^* = \prod_{b \in B_i} \mathbf{B}_{i,b}$.

Lemma 5. *Let \mathcal{A} be an instance of $\text{CSP}(\mathcal{B})$ and t be a binary term such that $t(x, t(x, y)) = t(x, y)$. If \mathcal{A} has a solution, then so does $t(\mathcal{A})$.*

Proof. Let f be a solution of the instance \mathcal{A} . We define a solution g of $t(\mathcal{A})$ as

$$g((i, b)) = t(b, f_i)$$

for all $(i, b) \in V'$. Clearly, $g((i, b)) \in B_{i,b}$. Take a scope $(I, r) \in S'$. By definition,

$$g|_{(I,r)} = \langle t(r_i, f_i) : i \in I \rangle = t(r, f|_I).$$

However, f is a solution, so both r and $f|_I$ are in \mathbf{R}_I and therefore $t(r, f|_I) \in \mathbf{R}_I$ as well. Clearly, $t(r, f|_I) \in \prod_{i \in I} \mathbf{B}_{i,r_i}$, thus we have shown that $g|_{(I,r)} \in \mathbf{R}_{i,r}$.

Now take a scope $(i, B_i) \in U'$ of the second kind. Here

$$g|_{(i,B_i)} = \langle t(b, f_i) : b \in B_i \rangle,$$

that is, it is one of the generating elements of \mathbf{T}_{i,B_i} . \square

In the next lemma we will try to understand the structure of the \mathbf{T}_{i,B_i} relations in $t(\mathcal{A})$, so we focus on a single $\mathbf{B} = \mathbf{B}_i$ algebra for a moment.

Lemma 6. *Let \mathbf{B} be an algebra, and t be a binary term such that $t(x, t(x, y)) = t(x, y)$. For $b \in B$ let $\mathbf{B}_b = t_b(\mathbf{B})$, and put $\mathbf{B}^* = \prod_{b \in B} \mathbf{B}_b$. Let*

$$\mathbf{T} = \text{Sg}_{\mathbf{B}^*} \{ \langle t(b, c) : b \in B \rangle \mid c \in B \}$$

and take a tuple $r \in T$. Then the following hold.

- (1) *The map $p : b \mapsto r_b$ is a unary polynomial of \mathbf{B} .*
- (2) *Let $b_1, b_2 \in B$ and ϑ be a congruence of \mathbf{B} . If $t(b_1, x) \equiv_{\vartheta} t(b_2, x)$ for all $x \in B$, then $p(b_1) \equiv_{\vartheta} p(b_2)$.*

Proof. Each generator tuple $\langle t(b, c) : b \in B \rangle$ of \mathbf{T} is actually a map from B to B and it is a unary polynomial \mathbf{B} in the variable b where c is a constant. When we generate the subalgebra by these vectors, then we take a basic operation f of \mathbf{B} , some unary polynomials $p_1(b), \dots, p_k(b)$ already generated and generate the tuple $p(b) = t(b, f(p_1(b), \dots, p_k(b)))$, which is again a unary polynomial of \mathbf{B} in the variable b .

To prove the second claim it is enough to see that $s(b_1) \equiv_{\vartheta} s(b_2)$ for each generator tuple s and verify that this property is preserved. So assume that the unary polynomials p_1, \dots, p_k are already generated and $p_1(b_1) \equiv_{\vartheta} p_1(b_2), \dots, p_k(b_1) \equiv_{\vartheta} p_k(b_2)$. Thus $c_1 = f(p_1(b_1), \dots, p_k(b_1)) \equiv_{\vartheta} f(p_1(b_2), \dots, p_k(b_2)) = c_2$, and using again our assumption that $t(b_1, x) \equiv_{\vartheta} t(b_2, x)$, we get that $p(b_1) = t(b_1, c_1) \equiv_{\vartheta} t(b_1, c_2) \equiv_{\vartheta} t(b_2, c_2) = p(b_2)$ for the newly generated polynomial p . \square

Lemma 7. *Let \mathcal{A} be an instance of $\text{CSP}(\mathcal{B})$ and t be a binary term such that $t(x, t(x, y)) = t(x, y)$. If $t(\mathcal{A})$ has a solution, then there exists a consistent set $\{p_i : i \in V\}$ of unary polynomials for the instance \mathcal{A} such that each polynomial p_i of \mathbf{B}_i satisfies the conclusion of Lemma ??.*

Proof. Let g be a solution of $t(\mathcal{A})$. We define a consistent set $p = \{p_i \mid i \in V\}$ of unary maps for \mathcal{A} as

$$p_i(b) = g((i, b))$$

for $i \in V$ and $b \in B_i$. By Lemma ??, each map $p_i : B_i \rightarrow B_i$ is a unary polynomial of \mathbf{B}_i . To see that it preserves the relations of \mathcal{A} take a scope $I \in S$ and a tuple $r \in \mathbf{R}_I$. Since g was a solution to $t(\mathcal{A})$ it respects the constraint $\mathbf{R}_{I,r}$, that is the tuple $\langle g((i, r_i)) : i \in I \rangle$ is in $\mathbf{R}_{I,r} \subseteq \mathbf{R}_I$. But this tuple is exactly $p|_I(r)$, which shows that p is consistent. \square

Definition 8. We say that an idempotent algebra \mathbf{B} can be *eliminated*, if whenever \mathcal{B} is a template such that $\mathbf{B} \in \mathcal{B}$, $\mathcal{B} \setminus \mathbf{B}$ is also a template, and $\text{CSP}(\mathcal{B} \setminus \{\mathbf{B}\})$ is tractable, then $\text{CSP}(\mathcal{B})$ is also tractable.

Lemma 9. *Let \mathbf{B} be an algebra and t be a binary term of \mathbf{B} such that for each $b \in B$ the map $t_b(x) = t(b, x)$ is idempotent and not surjective. Let C be the set of elements $c \in B$ such that $x \mapsto t(x, c)$ is a permutation. If C generates a proper subuniverse of \mathbf{B} , then \mathbf{B} can be eliminated.*

Proof. Let \mathcal{B} be a template containing \mathbf{B} and let \mathcal{A} be an instance of $\text{CSP}(\mathcal{B})$ containing at least one copy of \mathbf{B} . Replace all occurrence of \mathbf{B} in \mathcal{A} with the subalgebra generated by the set C . Clearly, this new instance is an instance of $\text{CSP}(\mathcal{B} \setminus \{\mathbf{B}\})$ so it can be solved in polynomial time. If it has a solution, then we are done, so we can assume that it does not.

Since the maps t_b are not surjective, $|t_b(\mathbf{B})| < |\mathbf{B}|$ and therefore the decomposition $t(\mathcal{A})$ is an instance of $\text{CSP}(\mathcal{B} \setminus \{\mathbf{B}\})$. Thus it can be solved in polynomial time. If $t(\mathcal{A})$ has no solution, then \mathcal{A} has no solution either by Lemma ??. On the other hand if $t(\mathcal{A})$ has a solution, then by Lemma ?? we have a consistent set $p = \{p_i : i \in V\}$ of unary polynomials for \mathcal{A} . Let us assume for a moment that p is not permutational. Now p can be iterated to obtain an idempotent non-permutational consistent set p' of unary polynomials for \mathcal{A} . By Lemma ?? we know

that \mathcal{A} has a solution if and only if $p'(\mathcal{A})$ does. Also, since p' is non-permutational, at least one of the domains of $p'(\mathcal{A})$ is smaller than that of \mathcal{A} . So by iterating this procedure we will get to a point when the algebra \mathbf{B} no longer occurs in the instance \mathcal{A} .

Now we go back to the problem of making sure that p becomes non-permutational. We know that if \mathcal{A} has a solution f , then for at least one $i \in V$, $\mathbf{B}_i = \mathbf{B}$ and $f_i \notin C$. Let us iterate through all variables $i \in V$ such that $B_i = B$ and all elements $d \in B \setminus C$. For each choice of i and d we create a new instance from $t(\mathcal{A})$ by adding new unary constraints stating that the solution $g|_{(i, B_i)} = \langle t(b, d) : b \in B_i \rangle$. This ensures that $p_i(b) = t(b, d)$, that is it is not permutational. If for any of these choices we find a non-permutational case, then we can reduce the instance as shown above. Otherwise we conclude that the instance has no solution. \square

2. APPLICATION

Corollary 10. *Let \mathbf{B} be an idempotent algebra, and $\beta \in \text{Con } \mathbf{B}$ such that \mathbf{B}/β is a semilattice (possibly with more operations) having more than one maximal element. Then \mathbf{B} can be eliminated.*

Proof. Take a binary term t of \mathbf{B} that is a semilattice operation on \mathbf{B}/β . We can assume, that $t(x, t(x, y)) = t(x, y)$ on \mathbf{B} . Since \mathbf{B}/β has more than one maximal element, for all $b \in B$ the maps $x \mapsto t(b, x)$ and $x \mapsto t(x, b)$ are not permutations. Thus we can apply Lemma ?? with $C = \emptyset$. \square

Corollary 11. *Let \mathbf{B} be an idempotent algebra, $\beta \in \text{Con } \mathbf{B} \setminus \{1_{\mathbf{B}}\}$ and t be a binary term such that t is a semilattice operation on \mathbf{B}/β . If the largest β -block (with respect to the semilattice order) contains more than one element and satisfies $t(x, y) = x$, then \mathbf{B} can be eliminated.*

Proof. We can assume that $t(x, t(x, y)) = t(x, y)$ on \mathbf{B} , since we can iterate t in the second variable without destroying the required properties stated in the lemma. By Corollary ??, \mathbf{B}/β has a largest element Q . Suppose, that the β -block Q has more than one element. Then the maps $t_b(x) = t(b, x)$ are not permutations. Moreover, for any $c \in B$ for which $x \mapsto t(x, c)$ is a permutation we must have $c \in Q$. However, Q is a proper subuniverse of \mathbf{B} , thus we can apply Lemma ?? to finish the proof. \square

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