A Convex-Regularization Framework for Local-Volatility Calibration in Derivative Markets

Jorge P. Zubelli

IMPA
Joint work with
A. De Cezaro (FURG,Brazil)
O. Scherzer (U.Vienna, Austria)

June 21, 2010

Outline

Motivation and Goals

Good model selection is crucial for modern sound financial practice.

Motivation and Goals

Good model selection is crucial for modern sound financial practice.

Focus: Dupire [Dup94] local volatility models

Good model selection is crucial for modern sound financial practice.

Focus: Dupire [Dup94] local volatility models

Goal:

- Present a unified framework for the calibration of local volatility models

Motivation and Goals

Good model selection is crucial for modern sound financial practice.

Focus: Dupire [Dup94] local volatility models

Goal:

- Present a unified framework for the calibration of local volatility models
- Use recent tools of convex regularization of ill-posed Inverse Problems.

Motivation and Goals

Good model selection is crucial for modern sound financial practice.

Focus: Dupire [Dup94] local volatility models

Goal:

- Present a unified framework for the calibration of local volatility models
- Use recent tools of convex regularization of ill-posed Inverse Problems.
- Present convergence results that include convergence rates w.r.t. noise level in fairly general contexts

Motivation and Goals

Good model selection is crucial for modern sound financial practice.

Focus: Dupire [Dup94] local volatility models

Goal:

- Present a unified framework for the calibration of local volatility models
- Use recent tools of convex regularization of ill-posed Inverse Problems.
- Present convergence results that include convergence rates w.r.t. noise level in fairly general contexts
- Go beyond the classical quadratic regularization.

Motivation and Goals

Good model selection is crucial for modern sound financial practice.

Focus: Dupire [Dup94] local volatility models

Goal:

- Present a unified framework for the calibration of local volatility models
- Use recent tools of convex regularization of ill-posed Inverse Problems.
- Present convergence results that include convergence rates w.r.t. noise level in fairly general contexts
- Go beyond the classical quadratic regularization.

Application

Volatility surface calibration is crucial in many applications. E.G.: risk management, hedging, and the evaluation of exotic derivatives.

Main Features

- Address in a general and rigorous way the key issue of convergence and sensitivity of the regularized solution when the noise level of the observed prices goes to zero.
- Our approach relates to different techniques in volatility surface estimation. e.g.: the Statistical concept of exponential families and entropy-based estimation.
- Our framework connects with the Financial concept of Convex Risk Measures.

Problem Statement

Starting Point: Dupire forward equation [Dup94]

$$-\partial_T U + \frac{1}{2}\sigma^2(T,K)K^2\partial_K^2 U - (r-q)K\partial_K U - qU = 0\,, \quad T > 0\,, \tag{1}$$

$$K = S_0 e^y\,,\ \tau = T - t\,,\ b = q - r\,, \quad u(\tau,y) = e^{q\tau}U^{t,S}(T,K) \tag{2}$$

and

$$a(\tau,y) = \frac{1}{2}\sigma^2(T-\tau; S_0 e^y)\,, \tag{3}$$

Set $q = r = 0$ for simplicity to get:

$$u_\tau = a(\tau,y)(\partial_y^2 u - \partial_y u) \tag{4}$$

and initial condition

$$u(0,y) = S_0(1-e^y)^+ \tag{5}$$

Problem Statement

The Vol Calibration Problem

Given an observed set

$$\{u = u(t, S, T, K; \sigma)\}_{(T,K)\in\mathcal{S}}$$

find $\sigma = \sigma(t, S)$ that best fits such market data

Noisy data: $u = u^\delta$

Admissible *convex* class of calibration parameters:

$$\mathcal{D}(F) := \{a \in a_0 + U : \underline{a} \le a \le \overline{a}\}. \qquad (6)$$

where, for $0 \le \varepsilon$ fixed, $U := H^{1+\varepsilon}(\Omega)$ and $\overline{a} > \underline{a} > 0$.

Parameter-to-solution operator

$F : \mathcal{D}(F) \subset U \to V$

$$F(a) = u(a) \qquad (7)$$

Literature

Very vast!!!

- Avellaneda et al. [ABF$^+$00, Ave98c, Ave98b, Ave98a, AFHS97]
- Bouchev & Isakov [BI97]
- Crepey [Cré03]
- Derman et al. [DKZ96]
- Hofmann et al. [HKPS07, HK05]
- Jermakyan [BJ99]
- Egger & Engl [EE05]
- Abken et al. (1996)
- Ait Sahalia, Y & Lo, A (1998)
- Berestycki et al. (2000)
- Buchen & Kelly (1996)
- Coleman et al. (1999)
- Cont, Cont & Da Fonseca (2001)
- Jackson et al. (1999)
- Jackwerth & Rubinstein (1998)
- Jourdain & Nguyen (2001)
- Lagnado & Osher (1997)
- Samperi (2001)
- Stutzer (1997)

Approach

Convex Tikhonov Regularization

For given convex f minimize the Tikhonov functional

$$\mathcal{F}_{\beta,u^\delta}(a) := ||F(a) - u^\delta||^2_{L^2(\Omega)} + \beta f(a) \tag{8}$$

over $\mathcal{D}(F)$, where, $\beta > 0$ is the regularization parameter.

Remark that f incorporates the *a priori* info on a.

Convex Tikhonov Regularization

For given convex f minimize the Tikhonov functional

$$\mathcal{F}_{\beta,u^\delta}(a) := ||F(a) - u^\delta||^2_{L^2(\Omega)} + \beta f(a) \tag{8}$$

over $\mathcal{D}(F)$, where, $\beta > 0$ is the regularization parameter.

Remark that f incorporates the *a priori* info on a.

$$||\bar{u} - u^\delta||_{L^2(\Omega)} \leq \delta\,, \tag{9}$$

where $\bar{u}$ is the data associated to the actual value $\hat{a} \in \mathcal{D}(F)$.

Approach

Convex Tikhonov Regularization

For given convex f minimize the Tikhonov functional

$$\mathcal{F}_{\beta,u^\delta}(a) := ||F(a) - u^\delta||^2_{L^2(\Omega)} + \beta f(a) \tag{8}$$

over $\mathcal{D}(F)$, where, $\beta > 0$ is the regularization parameter.

Remark that f incorporates the *a priori* info on a.

$$||\bar{u} - u^\delta||_{L^2(\Omega)} \leq \delta\,, \tag{9}$$

where $\bar{u}$ is the data associated to the actual value $\hat{a} \in \mathcal{D}(F)$.

Assumption (very general!)

Let $\varepsilon \geq 0$ be fixed. $f : \mathcal{D}(f) \subset U \longrightarrow [0, \infty]$ is a convex, proper and sequentially weakly lower semi-continuous functional with domain $\mathcal{D}(f)$ containing $\mathcal{D}(F)$.

Assumption (1)

1. *U and V given topologies τ_U and τ_V weaker than the norm topologies.*
2. *The norm $\|\cdot\|_V$ is sequentially lower semi-continuous w.r.t. τ_V.*
3. *The functional $f : \mathcal{D}(f) \subseteq U \to [0, \infty]$ is convex and sequentially lower semi-continuous w.r.t. τ_U and $\mathcal{D} := \mathcal{D}(F) \cap \mathcal{D}(f) \neq \emptyset$.*
4. *Let $\mathcal{F}_{\beta,\bar{u}}$ the Tikhonov functional defined in (8). Then,*

   $$\mathcal{M}_\beta(M) := level_M(\mathcal{F}_{\beta,\bar{u}}) = \{a : \mathcal{F}_{\beta,\bar{u}}(a) \leq M\}$$

   is sequentially pre-compact and closed w.r.t. τ_U.
5. *The restriction of F to $\mathcal{M}_\beta(M)$ are sequentially continuous w.r.t. the topologies τ_U and τ_V.*

Main Theoretical Result

$F(a) = u(a)$ (7) $\qquad \mathcal{F}_{\beta,u^\delta}(a) := ||F(a) - u^\delta||^2_{L^2(\Omega)} + \beta f(a)$ (8)

Theorem (Existence, Stability, Convergence)

Sup. F, f, $\mathcal{D}$, U, V satisfy Assumption 1, $\beta > 0$ and $u^\delta \in V$. Then,

- *$\exists$ minimizer of $\mathcal{F}_{\beta,u^\delta}$.*
- *If $(u_k) \to u$ in V w.r.t. norm topology, then (a_k) s.t.*

$$a_k \in \operatorname{argmin}\left\{\mathcal{F}_{\beta,u_k}(a) : a \in \mathcal{D}\right\}$$

  has a subsequence which converges w.r.t. τ_U.
- *The limit of every τ_U-convergent subsequence $(a_{k'})$ of (a_k) is a minimizer $\tilde{a}$ of $\mathcal{F}_{\beta,u}$, and $\left(f(a_{k'})\right)$ converges to $f(\tilde{a})$.*
- *If $\exists$ a solution of (7) in $\mathcal{D}$, then $\exists$ an f-minimizing solution of (7).*

$F(a) = u(a) \quad (7) \qquad \mathcal{F}_{\beta,u^\delta}(a) := ||F(a) - u^\delta||^2_{L^2(\Omega)} + \beta f(a) \quad (8)$

Theorem (cont.)

Take $\beta = \beta(\delta) > 0$ and assume

- (7) has a solution in $\mathcal{D}$
- $\beta(\delta)$ satisfies

$$\beta(\delta) \to 0 \text{ and } \frac{\delta^2}{\beta(\delta)} \to 0\,, \text{ as } \delta \to 0\,. \qquad (10)$$

- The seq. (δ_k) converges to 0, and that $u_k := u^{\delta_k}$ satisfies $\|\bar{u} - u_k\| \leq \delta_k$.

Then,

1. Every seq. $(a_k) \in \operatorname{argmin} \mathcal{F}_{\beta_k,u_k}$, has τ_U-convergent subseq. $(a_{k'})$.
2. The limit $a^\dagger := \lim_{\tau_U} a_{k'}$ is an f-minimizing solution of (7), and $f(a_k) \to f(a^\dagger)$.
3. If the f-minimizing solution $a^\dagger$ is unique, then $a_k \to a^\dagger$ w.r.t. τ_U.

Assumption 2

Bregman distance

Let f be a convex function. For $a \in \mathcal{D}(f)$, $\partial f(a) \subset U^*$ denotes the subdifferential of the functional f at a.

We denote by $\mathcal{D}(\partial f) = \{\tilde{a} : \partial f(\tilde{a}) \neq \emptyset\}$ the domain of the subdifferential.

The Bregman distance w.r.t $\zeta \in \partial f(a_1)$ is defined on $\mathcal{D}(f) \times \mathcal{D}(\partial f)$ by

$$D_\zeta(a_2, a_1) = f(a_2) - f(a_1) - \langle \zeta, a_2 - a_1 \rangle \; .$$

Assumption 2

Bregman distance

Let f be a convex function. For $a \in \mathcal{D}(f)$, $\partial f(a) \subset U^*$ denotes the subdifferential of the functional f at a.
We denote by $\mathcal{D}(\partial f) = \{\tilde{a} : \partial f(\tilde{a}) \neq \emptyset\}$ the domain of the subdifferential.
The Bregman distance w.r.t $\zeta \in \partial f(a_1)$ is defined on $\mathcal{D}(f) \times \mathcal{D}(\partial f)$ by

$$D_\zeta(a_2, a_1) = f(a_2) - f(a_1) - \langle \zeta, a_2 - a_1 \rangle \ .$$

Assumption (2)

Besides Assumption 1, we assume that

1. $\exists$ *an f-minimizing sol.* $a^\dagger$ *of* (7), $a^\dagger \in \mathcal{D}_B(f)$.
2. $\exists \beta_1 \in [0,1)$, $\beta_2 \geq 0$, *and* $\zeta^\dagger \in \partial f(a^\dagger)$ *s.t.*

$$\langle \zeta^\dagger, a^\dagger - a \rangle \leq \beta_1 D_{\zeta^\dagger}(a, a^\dagger) + \beta_2 \left\| F(a) - F(a^\dagger) \right\|_V \ \text{for } a \in \mathcal{M}_{\beta_{max}}(\rho) \ , \quad (11)$$

where $\rho > \beta_{max} f(a^\dagger) > 0$.

Convergence rates [SGG$^+$08]

Theorem (Convergence rates [SGG$^+$08])

Let F, f, $\mathcal{D}$, U, and V satisfy Assumption 2. Moreover, let $\beta : (0,\infty) \to (0,\infty)$ satisfy $\beta(\delta) \sim \delta$. Then

$$D_{\zeta^\dagger}(a^\delta_\beta, a^\dagger) = O(\delta)\,, \quad \left\| F(a^\delta_\beta) - u^\delta \right\|_V = O(\delta)\,,$$

and there exists $c > 0$, such that $f(a^\delta_\beta) \le f(a^\dagger) + \delta/c$ for every δ with $\beta(\delta) \le \beta_{max}$.

Putting it all together

Although Assumption 1 may seem too restrictive, the next result reveals that it can be obtained from rather classical ones:

Proposition

Let F, f, $\mathcal{D}$, U, and V satisfy Assumption 1. Assume that $\exists$ an f-minimizing solution $a^\dagger$ of (7), and that F is Gâteaux differentiable at $a^\dagger$.
Moreover, assume that $\exists \gamma \geq 0$ and $\omega^\dagger \in V^*$ with $\gamma \left\| \omega^\dagger \right\| < 1$, s.t.

$$\zeta^\dagger := F'(a^\dagger)^* \omega^\dagger \in \partial f(a^\dagger) \tag{12}$$

and $\exists \beta_{max} > 0$ satisfying $\rho > \beta_{max} f(a^\dagger)$ such that

$$\left\| F(a) - F(a^\dagger) - F'(a^\dagger)(a - a^\dagger) \right\| \leq \gamma D_{\zeta^\dagger}(a, a^\dagger), \text{ for } a \in \mathcal{M}_{\beta_{max}}(\rho) . \tag{13}$$

Then, Assumption 2 holds.

NOTE: We have proved

- The above hypothesis hold for the problem under consideration.
- We have proved a tangential cone condition, which implies that the Landwever iteration converges in a suitable neighborhood.

Landweber Iteration [EHN96]:

$$a_{k+1}^{\delta} = a_k^{\delta} + cF'(a_k^{\delta})^*(u^{\delta} - F(a_k^{\delta}))\,. \tag{14}$$

NOTE: We have proved

- The above hypothesis hold for the problem under consideration.
- We have proved a tangential cone condition, which implies that the Landwever iteration converges in a suitable neighborhood.

Landweber Iteration [EHN96]:

$$a_{k+1}^{\delta} = a_k^{\delta} + cF'(a_k^{\delta})^*(u^{\delta} - F(a_k^{\delta}))\,. \tag{14}$$

Discrepancy Principle:

$$\left\| u^{\delta} - F(a^{\delta}_{k_*(\delta,y^{\delta})}) \right\| \leq r\delta < \left\| u^{\delta} - F(a_k^{\delta}) \right\|\,, \tag{15}$$

where

$$r > 2\frac{1+\eta}{1-2\eta}\,, \tag{16}$$

is a relaxation term.
If the iteration is stopped at index $k_*(\delta, y^{\delta})$ such that for the first time, the residual becomes small compared to the quantity $r\delta$.

Connection with Statistics and Exponential Families

Regular Exponential Families:

family of probability distribution functions $p_{\psi,\theta} : \mathbb{R} \to \mathbb{R}_+$ defined by

$$p_{\psi,\theta}(s) := \exp(s \cdot \theta - \psi(\theta)) p_0(s)$$

where $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is convex and $p_0 : \mathbb{R} \to \mathbb{R}_+$ is continuous.

Example:

Gaussians parametrized by the mean.

The Darmois-Koopman-Pitman Thm: Under certain regularity conditions on the probability density, a necessary and sufficient condition for the existence of a sufficient statistic of fixed dimension is that the probability density belongs to the exponential family [And70].

Connection with Statistics and Exponential Families(cont.)

Recall the Fenchel Conjugate

Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, the *Fenchel* dual $f^* X^* \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) \mid x \in X\}$$

Theorem (Banerjee et al. [BMDG05])

Let ψ^* denote the Fenchel transform of ψ, which we assume to be differentiable. Then, the Bregman distance w.r.t. ψ^* is given by

$$D_{\psi^*}(\hat{a}, \tilde{a}) = \psi^*(\hat{a}) - \psi^*(\tilde{a}) - \psi^{*\prime}(\tilde{a})(\hat{a} - \tilde{a}) \,.$$

If we assume that $a(\theta) \in int(dom(\psi^*))$, then

$$p_{\psi,\theta}(a) = \exp\big(-D_{\psi^*}(a, a(\theta))\big)\exp\big(\psi^*(a)\big)p_0(a) \,. \tag{17}$$

Connection with Statistics and Exponential Families(cont.)

Example (Exponential Families and their Fenchel conjugates)

For a Gaussian distribution $\psi(\theta) = \frac{\varpi^2}{2}\theta^2$, *then* $\psi^*(a) = \frac{a^2}{2\varpi^2}$. *For Poisson distribution* $\psi(\theta) = \exp(\theta)$ *we have* $\psi^*(a) = a\log(a) - a$.

Example

According to Example 1, if we use the exponential family associated to Poisson distributions, we obtain Kullback-Leibler regularization, consisting in minimization of

$$a \longmapsto \mathcal{F}_{\beta,u^\delta}(a) := \left\| F(a) - u^\delta \right\|^2_{L^2(\Omega)} + \beta KL(\hat{a}, a) \,, \tag{18}$$

where

$$KL(\hat{a}, a) = \int_\Omega a\log(\hat{a}/a) - (\hat{a} - a)\, dx \quad .$$

We note that the Kullback-Leibler distance is the Bregman distance associated to the Boltzmann-Shannon entropy

Connection with Statistics and Exponential Families(cont.)

Lemma

Let Ω be a bounded subset of $\mathbb{R}^2$ with Lipschitz boundary. Moreover, assume that F is continuous w.r.t. the weak topologies on $L^1(\Omega)$ and $L^2(\Omega)$, respec.

1. *Let $a, b \in \mathcal{D}(\mathcal{G})$. Then*

$$\|a-b\|^2_{L^1(\Omega)} \leq \left(\frac{2}{3}\|a\|_{L^1(\Omega)} + \frac{4}{3}\|b\|_{L^1(\Omega)}\right) KL(a,b)\,. \tag{20}$$

   (Convention: $0 \cdot (+\infty) = 0$)

2. *Let $0 \neq \hat{a} \in \mathcal{D}_B(\mathcal{G})$, then the sets*

$$\mathcal{M}_{\beta,u^\delta}(M) := \{a \in \mathcal{D}_B(\mathcal{G}) : \mathcal{F}_{\beta,u^\delta}(a) \leq M\}$$

   are $\tau_{\tilde{U}}$ sequentially compact.

An important consequence of (20) and Theorem 2 is that

$$\left\| a_{\beta}^{\delta} - a^{\dagger} \right\|_{L^1(\Omega)} = \mathcal{O}(\sqrt{\delta}) \,. \tag{21}$$

Now, let δ_k be a sequence converging to zero and $a_k = a_{\beta_k}^{\delta_k}$ the respective minimizers of the Tikhonov functional (8). Take $b_k = a^{\dagger}$ for all $k \in \mathbb{N}$. Then, from Lemma 1

$$\left\| a_k - a^{\dagger} \right\|_{L^1(\Omega)} \to 0 \,, \quad as \quad \delta_k \to 0 \,.$$

Connection w/ Convex Risk Measures

Convex measure of risk

Consists of a map $\rho : \mathcal{X} \longrightarrow \mathbb{R}$ satisfying the following properties:

- Convexity.
- Non-increasing monotonicity, i.e., $\nu_2 \leq \nu_1$ a.e. implies $\rho(\nu_2) \geq \rho(\nu_1)$.
- Translation invariance, i.e., $m \in \mathbb{R}$ deterministic implies

$$\rho(\nu + m) = \rho(\nu) - m. \qquad (22)$$

We assume that the domain $\Omega = [0, T] \times I$

Theorem

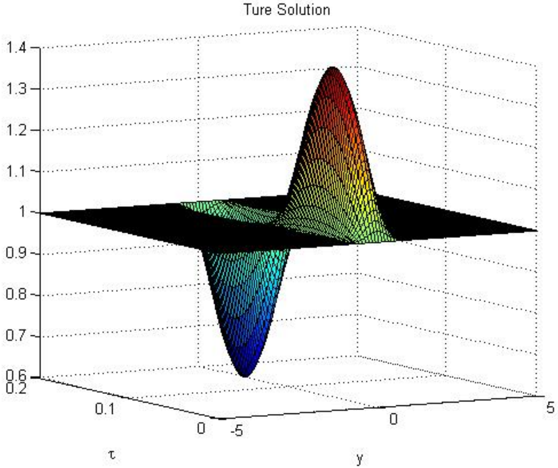
The source condition (12)

$$\zeta^{\dagger} := F'(a^{\dagger})^{*}\omega^{\dagger} \in \partial f(a^{\dagger}) \qquad (12)$$

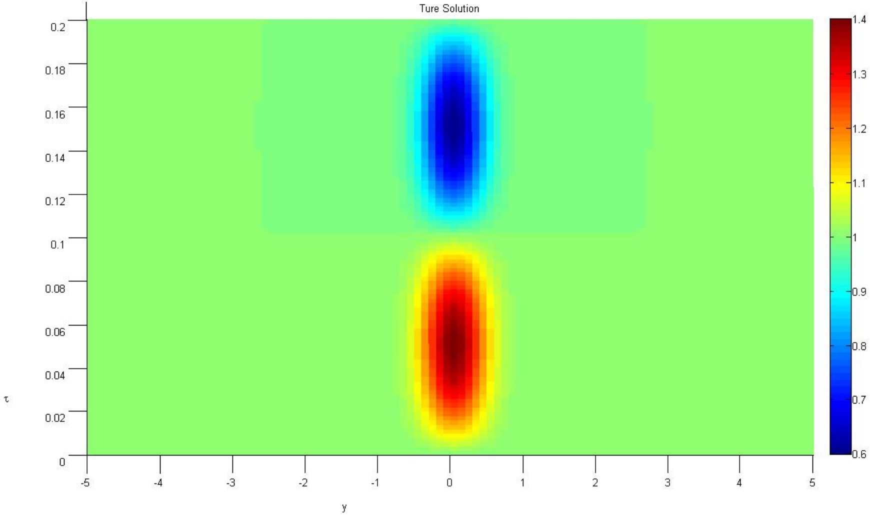
can be interpreted as an a priori assumption on the risk associated to the correspondent position, given the volatility level.

- Using a Landweber iteration technique we implemented the calibration.
- Produced for different test variances a the option prices and added different levels of multiplicative noise.
- The examples consisted of perturbing $a = 1$ during a period of $T = 0, \cdots, 0.2$ and log-moneyness y varying between -5 and 5.
- Initial guess: Constant volatility.

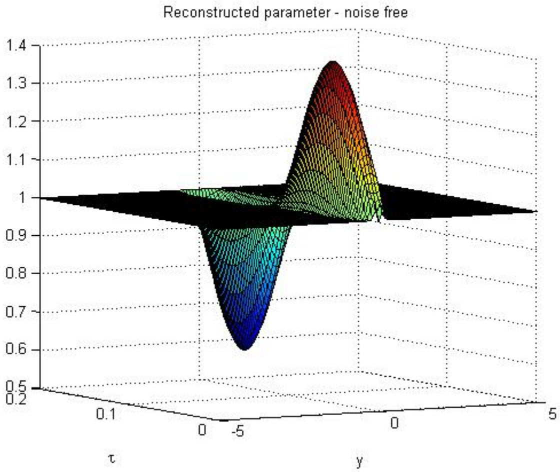
Numerical Examples - Exact Solution



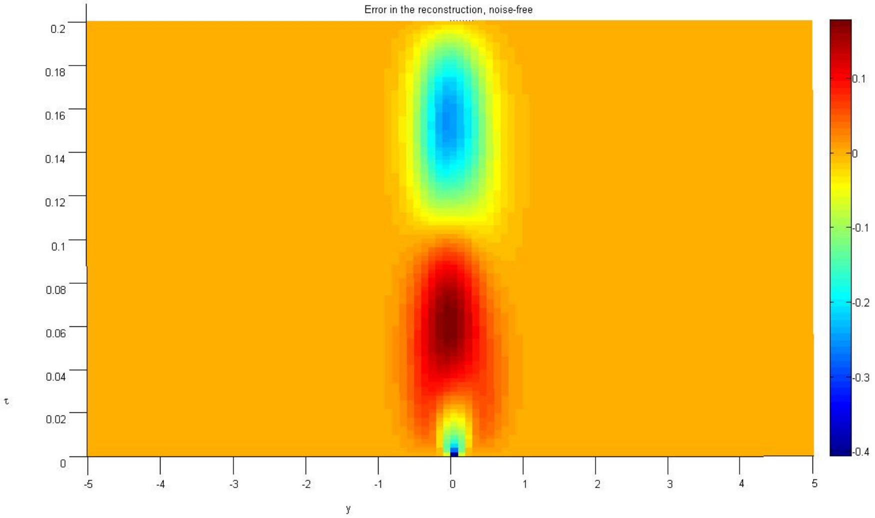
Numerical Examples - Exact Solution



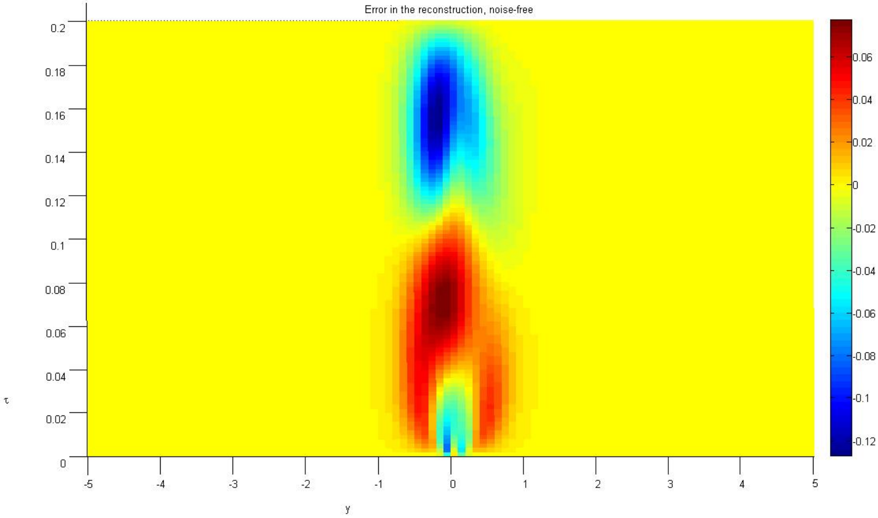
Numerical Examples 1 - noiseless - 4000 steps



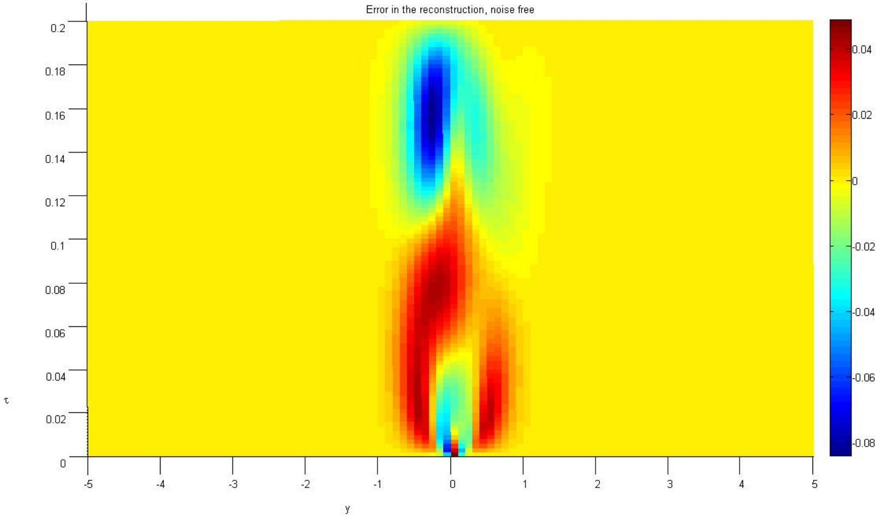
Numerical Examples 1 - error - 100 steps



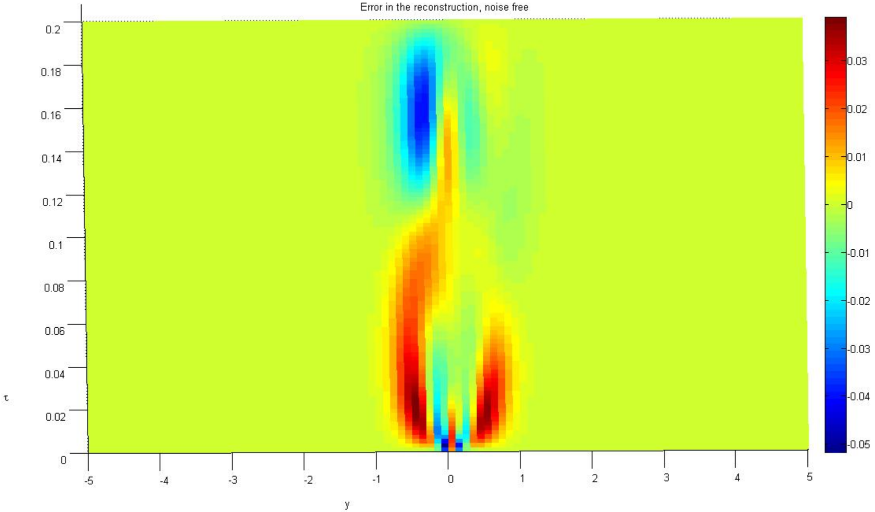
Numerical Examples 1 - error - 300 steps



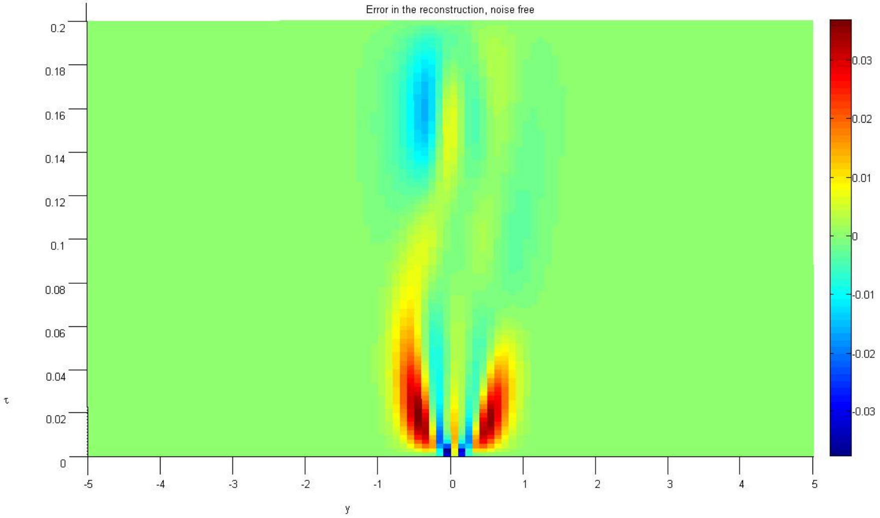
Numerical Examples 1 - error - 500 steps



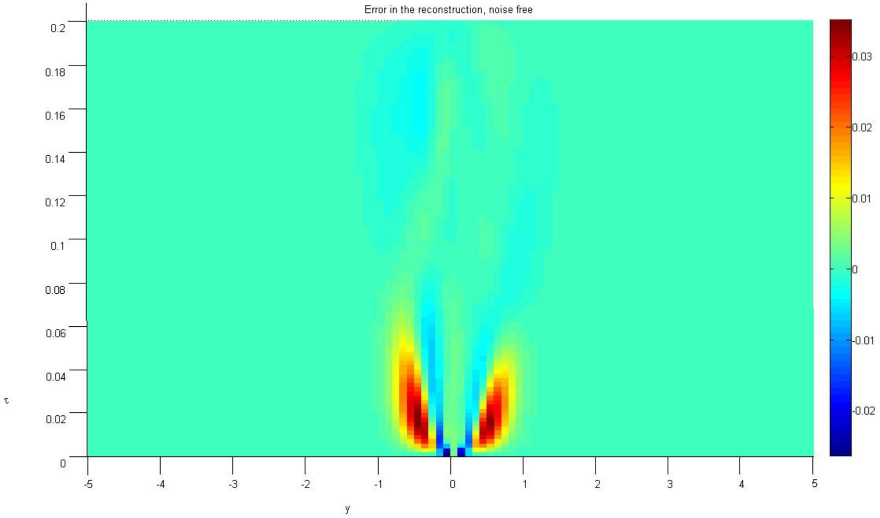
Numerical Examples 1 - error - 1000 steps

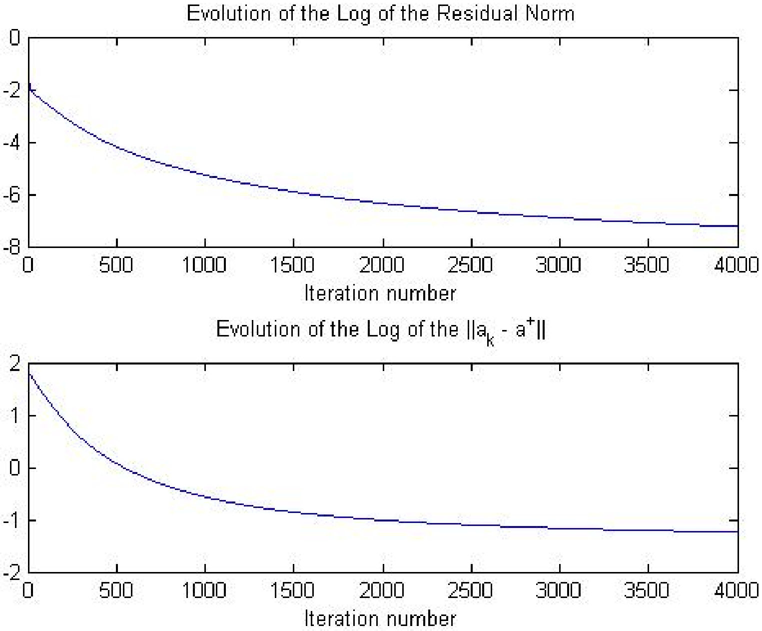


Numerical Examples 1 - error - 2000 steps

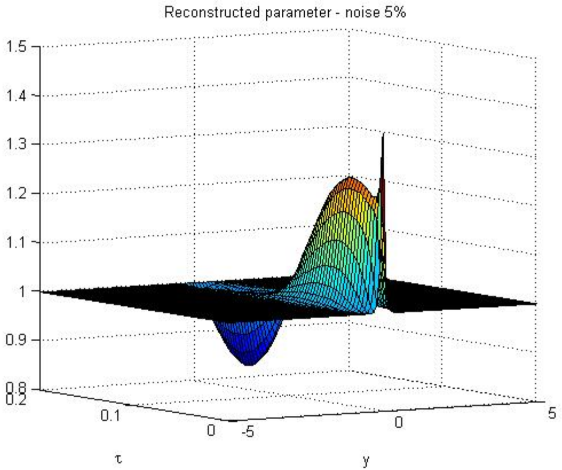


Numerical Examples 1 - error - 4000 steps

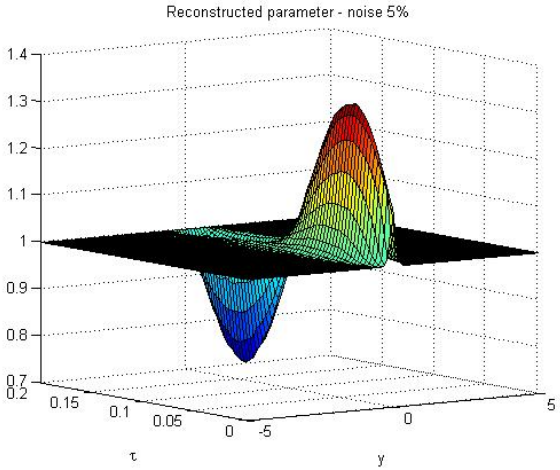



Evolution of the Log of the Residual Norm
0
-2
-4
-6
-8
0 500 1000 1500 2000 2500 3000 3500 4000
Iteration number
Evolution of the Log of the ||a_k - a^+||
2
1
0
-1
-2
0 500 1000 1500 2000 2500 3000 3500 4000
Iteration number

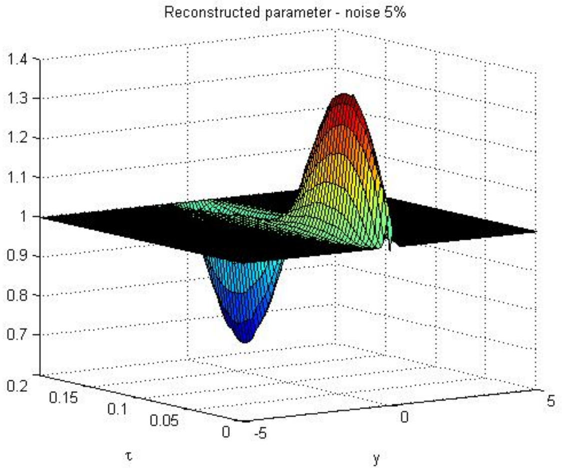
Numerical Examples 2 - 5% noise level - 100 steps



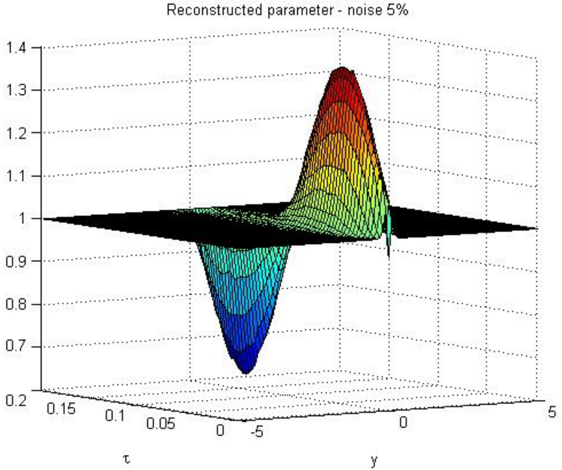
Numerical Examples 2 - 5% noise level - 200 steps



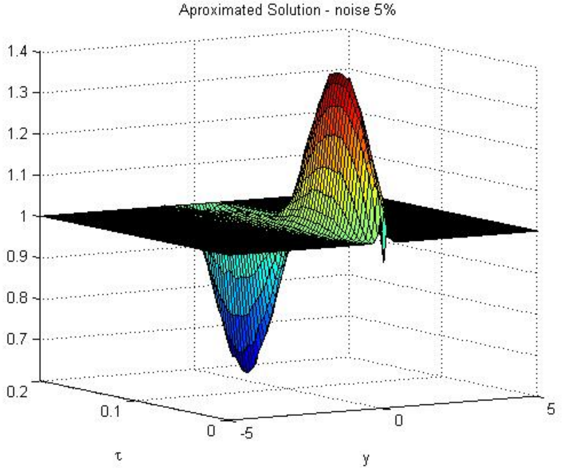
Numerical Examples 2 - 5% noise level - 300 steps

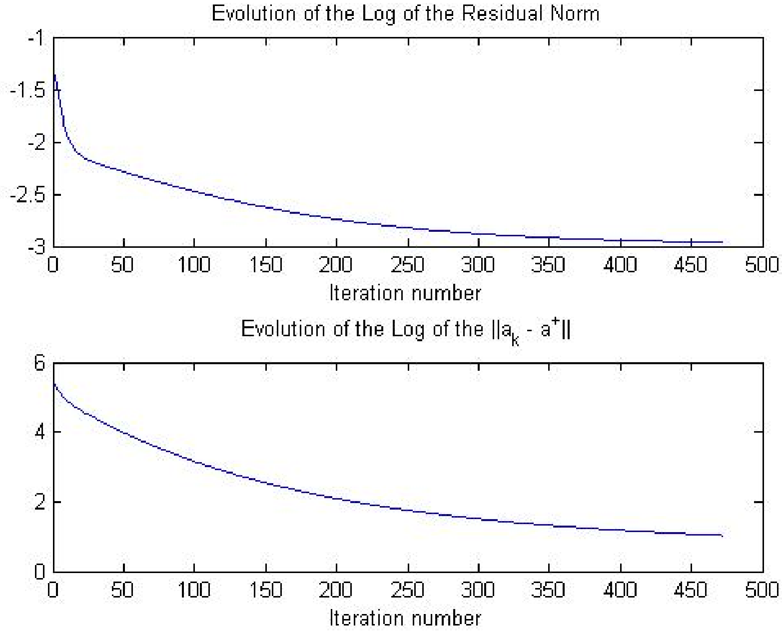


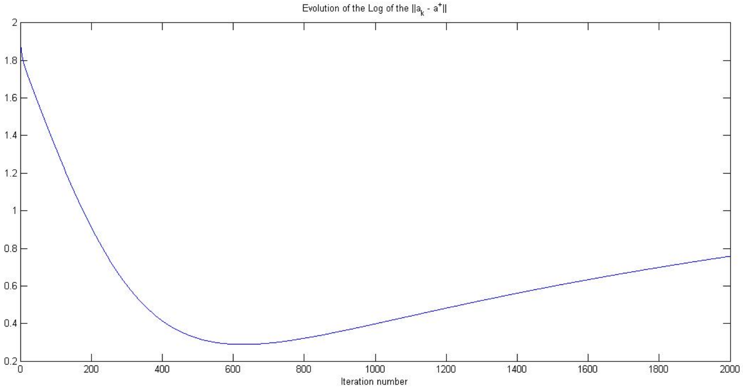
Numerical Examples 2 - 5% noise level - 400 steps



Numerical Examples 2 - 5% noise level - Stopping criteria

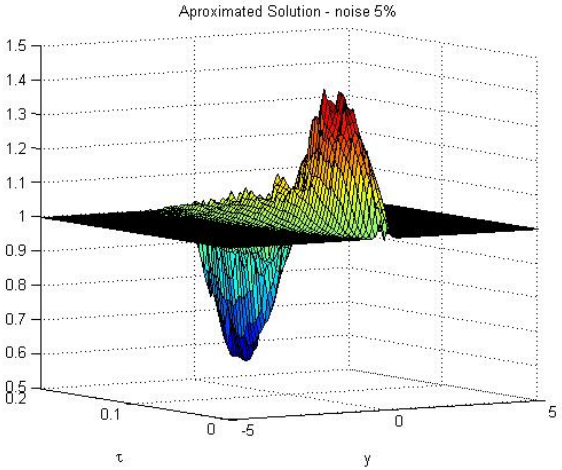



Evolution of the Log of the Residual Norm
-1
-1.5
-2
-2.5
-3
0
50
100
150
200
250
300
350
400
450
500
Iteration number
Evolution of the Log of the ||a_k - a^+||
6
4
2
0
0
50
100
150
200
250
300
350
400
450
500
Iteration number


Evolution of the Log of the ||a_k - a^+||
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
200
400
600
800
1000
1200
1400
1600
1800
2000
Iteration number

Numerical Examples 2 - 5% noise level - 2000 iterations

Too many!!!



Conclusions

- The problem of volatility surface calibration is a classical and fundamental one in Quantitative Finance
- Unifying framework for the regularization that makes use of tools from Inverse Problem theory and Convex Analysis.
- Establishing convergence and convergence-rate results.
- Obtain convergence of the regularized solution with respect to the noise level in $L^1(\Omega)$
- The connection with exponential families opens the door to recent works on entropy-based estimation methods.
- The connection with convex risk measures required the use of techniques from Malliavin calculus.
- Implemented a Landweber type calibration algorithm.

.

M. Avellaneda, R. Buff, C. Friedman, N. Grandchamp, L. Kruk, and J. Newman.
Weighted Monte Carlo: A new technique for calibrating asset-pricing models.
Spigler, Renato (ed.), Applied and industrial mathematics, Venice-2, 1998. Selected papers from the 'Venice-2/Symposium', Venice, Italy, June 11-16, 1998. Dordrecht: Kluwer Academic Publishers. 1-31 (2000)., 2000.

M. Avellaneda, C. Friedman, R. Holmes, and D. Samperi.
Calibrating volatility surfaces via relative-entropy minimization.
Appl. Math. Finance, 4(1):37–64, 1997.

E. B. Andersen.
Sufficiency and exponential families for discrete sample spaces.
65:1248–1255, 1970.

M. Avellaneda.
Minimum-relative-entropy calibration of asset-pricing models.
International Journal of Theoretical and Applied Finance, 1(4):447–472, 1998.

Marco Avellaneda.
The minimum-entropy algorithm and related methods for calibrating asset-pricing model.
In *Trois applications des mathématiques*, volume 1998 of *SMF Journ. Annu.*, pages 51–86. Soc. Math. France, Paris, 1998.

Marco Avellaneda.
The minimum-entropy algorithm and related methods for calibrating asset-pricing models.
In *Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998)*, number Extra Vol. III, pages 545–563 (electronic), 1998.

I. Bouchouev and V. Isakov.
The inverse problem of option pricing.
Inverse Problems, 13(5):L11–L17, 1997.

James N. Bodurtha, Jr. and Martin Jermakyan.
Nonparametric estimation of an implied volatility surface.
Journal of Computational Finance, 2(4), Summer 1999.

A. Banerjee, S. Merugu, I.S. Dhillon, and J. Ghosh.

Clustering with bregman divergences.
Journal of Machine Learning Research, 6:1705–1749, 2005.

S. Crépey.
Calibration of the local volatility in a generalized Black-Scholes model using Tikhonov regularization.
SIAM J. Math. Anal., 34(5):1183–1206 (electronic), 2003.

Emanuel Derman, Iraj Kani, and Joseph Z. Zou.
The local volatility surface: Unlocking the information in index option prices.
Financial Analysts Journal, 52(4):25–36, 1996.

B. Dupire.
Pricing with a smile.
Risk, 7:18– 20, 1994.

H. Egger and H. W. Engl.
Tikhonov regularization applied to the inverse problem of option pricing: convergence analysis and rates.
Inverse Problems, 21(3):1027–1045, 2005.

H. W. Engl, M. Hanke, and A. Neubauer.
Regularization of inverse problems, volume 375 of *Mathematics and its Applications*.
Kluwer Academic Publishers Group, Dordrecht, 1996.

B. Hofmann and R. Krämer.
On maximum entropy regularization for a specific inverse problem of option pricing.
J. Inverse Ill-Posed Probl., 13(1):41–63, 2005.

B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer.
A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators.
Inverse Problems, 23(3):987–1010, 2007.

O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier, and F. Lenzen.
Variational Methods in Imaging, volume 167 of *Applied Mathematical Sciences*.
Springer, New York, 2008.

THANK YOU!