Control Improvement for Jump-Diffusion Processes with Applications to Finance

Nicole Bäuerle
joint work with Ulrich Rieder

Toronto, June 2010

Karlsruhe Institute of Technology

Outline

- Motivation: MDPs
- Controlled Jump-Diffusion Processes
- Control Improvement Algorithm
- Financial Applications

Markov Decision Processes

Let (X_n) be a controlled Markov process with

- state space S, action space A,
- transition kernel $Q(\cdot|x,a)$.

Let $f : S \to A$ be a decision rule and

- $\beta \in (0,1)$ a discount factor,
- $r(x,a)$ a bounded reward function.

Consider the infinite-horizon Markov Decision Problem

$$J(x) := \sup_{f \in F} J_f(x) = \sup_{f \in F} \mathbb{E}_x \left[\sum_{n=0}^{\infty} \beta^n r(X_n, f(X_n)) \right].$$

Notation

- $B := \{v : S \to \mathbb{R} : \|v\|_\infty < \infty\}$.
- For $v \in B$ and $f : S \to A$ let

$$\mathcal{T}_f v(x) := r\big(x, f(x)\big) + \beta \int v(x')Q\big(dx'|x, f(x)\big).$$

- f^* is called *maximizer* of v if

$$\mathcal{T}_{f^*} v = \sup_{f \in F} \mathcal{T}_f v.$$

It holds that $J_f = \mathcal{T}_f J_f$ and $J = \sup_f \mathcal{T}_f J$.

Howard's Policy Improvement Algorithm

1. Choose f_0 arbitrary and set $k = 0$.
2. Compute J_{f_k} as solution $v \in B$ of the equation $v = \mathcal{T}_{f_k} v$.
3. Compute f_{k+1} as a maximizer of J_{f_k}.
   Then $J_{f_{k+1}} \geq J_{f_k}$. If $f_{k+1} = f_k$ then $J_{f_k} = J$ and $(f_k, f_k, \ldots)$ is optimal. Else set $k := k + 1$ and go to step 2.

Controlled Jump-Diffusion Processes

- $W = (W_1, \dots, W_m)$ is an m-dimensional Brownian motion,
- $N = (N_1, \dots, N_l)$ are indep. Poisson random measures,
- $\nu_j(B) := \mathbb{E}\, N_j(1, B)$ are the Lévy measures,
- $\tilde{N}_j(dt, dz_j) := N_j(dt, dz_j) - \nu_j(dz_j)dt$.

The n-dimensional controlled state process $X = (X_1, \dots, X_n)$ is

$$dX_i(t) = \mu_i(t, X_t, \pi_t)dt + \sum_{j=1}^{m} \sigma_{ij}(t, X_t, \pi_t)dW_j(t) + \sum_{j=1}^{l} \int \gamma_{ij}(t, X_{t-}, \pi_{t-}, z_j)\tilde{N}_j(dt, dz_j)$$

Controlled Jump-Diffusion Processes

- $\pi = (\pi_t)$ is a càdlàg control process with values in $D \subset \mathbb{R}^d$,
- the coefficient functions μ, σ, γ are continuous,
- g, h are reward functions.

Consider the problem

$$J^\pi(t,x) := \mathbb{E}_{t,x}\left[\int_t^T g(s, X_s, \pi_s)ds + h(X_T)\right].$$

$$J(t,x) = \sup_\pi J^\pi(t,x).$$

Generator of the state process

$$
\begin{aligned}
\mathcal{A}v(t,x,u) &= v_t(t,x) + \sum_{i=1}^{n} v_{x_i}(t,x)\mu_i(t,x,u) + \\
&+ \frac{1}{2}\sum_{i,j=1}^{n} (\sigma\sigma^T)_{ij}(t,x,u) v_{x_i x_j}(t,x) + \\
&+ \sum_{j=1}^{l} \int \Big(v(t, x + \gamma^{(j)}(t,x,u,z_j)) - v(t,x) - \\
&\qquad \nabla_x v(t,x)\gamma^{(j)}(t,x,u,z_j) \Big) \nu_j(dz_j).
\end{aligned}
$$

Control Improvement Algorithm

1. Suppose π^0 is an admissible control.
2. Compute the corresponding value function J^0 and suppose $J^0 \in C^{1,2}$.
3. Compute $\pi_1(t, x)$ such that it maximizes

$$u \mapsto g(t, x, u) + \mathcal{A}J^0(t, x, u), \quad u \in D$$

and suppose that $\pi_t^1 := \pi_1(t, X_t^1)$ is an admissible control.

Control Improvement Algorithm

Under some technical conditions it holds:

Theorem

Let $I := \{(t, x) : g(t, x, \pi_1(t, x)) + \mathcal{A}J^0(t, x, \pi_1(t, x)) > 0\}$.

a) *If* $I \neq \emptyset$, *then* $J^1(t, x) \geq J^0(t, x)$ *for all* (t, x) *and* $J^1(t, x) > J^0(t, x)$ *for* $(t, x) \in I$.

b) *If* $I = \emptyset$ *then* π^1 *is an optimal control.*

Limit Considerations

Theorem

Suppose that the following assumptions are satisfied:

(i) $\lim_{k\to\infty} J^k =: J^\infty \in C^{1,2}$ *and*
$J_t^k \to J_t^\infty, J_x^k \to J_x^\infty, J_{xx}^k \to J_{xx}^\infty$ *uniformly.*

(ii) μ, σ, γ *are bounded.*

Let π *be a policy defined by the maximizer of* J^∞ *as in step (b) of the algorithm, then* $J = J^\infty$ *and* π *is optimal.*

Financial Market

- The price process (S_t^0) of the riskless bond is given by

$$S_t^0 := e^{rt},$$

where $r \geq 0$ denotes the fixed continuous interest rate.

- The price process (S_t) of the risky asset satisfies:

$$dS_t = S_{t-}\left(\mu dt + \sigma dW_t + \int_{-1}^{\infty} z\tilde{N}(dt, dz)\right)$$

where $\mu \in \mathbb{R}, \sigma > 0$ and $\int_{-1}^{\infty} z\nu(dz) < \infty$.

- Øksendal and Sulem (2005)

Portfolio Optimization

- $U : (0, \infty) \to \mathbb{R}$ is a (strictly increasing, concave) utility function.
- (π_t) with $\pi_t \in [0, 1]$ is the portfolio strategy where π_t = fraction of wealth invested in the stock at time t.

The dynamics of the wealth process is

$$dX_t^\pi = X_t^\pi \left(rdt + \pi_t \cdot (\mu - r)dt + \pi_t \sigma dW_t + \pi_{t-} \int_{-1}^{\infty} z\tilde{N}(dt, dz) \right).$$

The portfolio problem is

$$J(t, x) := \sup_\pi \mathbb{E}[U(X_T^\pi) | X_t^\pi = x].$$

When is the "invest all the money in the bond"-strategy optimal?

Theorem

Let $U \in C^2(0, \infty)$ be an arbitrary utility function. The "invest all the money in the bond"-strategy is optimal if and only if $\mu \leq r$.

Proof

Consider $\pi_t \equiv 0$ with $J^\pi(t,x) = U(xe^{r(T-t)})$.
$\pi^* \equiv 0$ is again a maximum point of $u \mapsto \mathcal{A}J^\pi(t,x,u)$ on $[0,1]$ if and only if

$$\frac{\partial}{\partial u}\mathcal{A}J^\pi(t,x,u)|_{u=0} \quad = \quad (\mu - r)xJ^\pi_x \leq 0.$$

Special Case: Black-Scholes Model

Suppose now we have a Black-Scholes market. In case $\mu > r$, the first improvement of the ”invest all the money in the bond”-strategy is given by

$$\pi_1(t, x) = -\frac{U'(xe^{r(T-t)})}{U''(xe^{r(T-t)})xe^{r(T-t)}} \cdot \frac{(\mu - r)}{\sigma^2}.$$

It relies on the Arrow-Pratt-Relative-Risk-Aversion Coefficient and the Merton-ratio. When the utility function is the power or logarithmic utility function, the first improvement yields already the optimal investment strategy.

When is a constant fraction optimal?

Suppose ν is concentrated on $(0, \infty)$, i.e. jumps are only upwards and that $2 \int x\nu(dx) < \mu - r$. Under these assumptions it holds:

Theorem

The logarithm- and the power-utility are the only utility functions $U \in C^2(0, \infty)$ with $U \in C^2$ (up to a multiplicative constant) where the optimal portfolio invests a constant positive fraction of the wealth in the stock.

Proof

J^π and $\pi_t \equiv \pi$ are optimal if and only if π is a maximum point of $u \mapsto \mathcal{A}J^\pi(t,x,u), u \geq 0$, i.e.

$$(\mu - r)J_x^\pi + J_{xx}^\pi \sigma^2 x\pi + \int_0^\infty \Big(J_x^\pi(t, x + \pi xz)z - J_x^\pi(t,x) \Big) \nu(dz) = 0$$

and we must have $\mathcal{A}J^\pi(t,x,\pi) = 0$, i.e.

$$J_t^\pi + (r + (\mu - r)\pi)xJ_x^\pi + \frac{1}{2}J_{xx}^\pi \sigma^2 x^2 \pi^2 + \\ + \int_0^\infty \Big(J^\pi(t, x + \pi xz) - J^\pi(t,x) - J_x^\pi(t,x)\pi xz \Big) \nu(dz) = 0.$$

- Bäuerle, N., Rieder, U. (2010) : Control improvement for jump-diffusion processes with applications to finance. Preprint.
- Bäuerle, N., Rieder, U. (2010) : Markov Decision Processes with Applications to Finance. To appear.
- Fleming, W. H., Rishel, R. (1975) : Deterministic and stochastic optimal control. Springer-Verlag.
- Howard, R. (1960) : Dynamic programming and Markov processes. The Technology Press of M.I.T., Cambridge, Mass.
- Øksendal, B,,Sulem, A. (2005) : Applied stochastic control of jump diffusions. Springer-Verlag

Thank you very much
for your attention!