Spectral Capital Allocation and Applications

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BFS 2010 , Toronto

June, 2010

Agenda

- Spectral risk measure
- Axiomatic Capital Allocation
- Spectral Capital Allocation
- Examples

Spectral risk measures and allocation

A risk measure is a real valued function r defined on a the set of random variable (potential losses) V. The number r(X) denotes the risk in portfolio X. The following 4 properties are meanwhile well-known.

1. Subadditivity (Diversification)

$$r(X+Y) < r(X) + r(Y)$$

2. Positive homogenous (Scaling)

$$r(aX) = ar(X), a > 0$$

3. Monotone

$$r(X) < r(Y) \text{ if } X < Y \text{ (almost surely)}$$

4. Translation property

$$r(X+a) = r(X) - a$$

r is called coherent iff all 4 are satisfied (Artzner et al (1997, 1999), Delbaen (2000)).

Axioms 1 and 2 already lead to the representation (Delbaen (2000), Kalkbrener (2002), Kalkbrener et al (2004)) as a generalized gradient

$$r(X) \;=\; \max\{l(X) \mid l < r,\; l \;\text{ linear function }\} \tag{1}$$

Generalized Scenarios

The basic representation by Artzner et al (1997), Delbaen (2000) for coherent risk measures is formulated in terms of generalized scenarios

$$r(X) \ = \ \sup\{E_Q[X] \mid Q \in \mathcal{Q}\} \tag{2}$$

$\mathcal{Q}, = \mathcal{Q}_r$, a suitable set of probability measures of absolutely continuous probability measures $Q << P$ with density dQ/dP , is similar to the representation (**??**). If the supremum is indeed a maximum a density L_r is found and

$$r(X) = E[L_r X]$$

a very useful result if it comes to allocation.

Expected Shortfall

Expected Shortfall at level α is properly defined by (cf. Acerbi and Tasche (2002), Kalkbrener et al (2004):

Definition 1

$$\mathrm{ES}_\alpha(L) := (1-\alpha)^{-1}[\mathrm{E}(L\,\mathbf{1}_{\{L>\mathrm{q}_\alpha(L)\}}) + \mathrm{q}_\alpha(L)\cdot(\mathbb{P}(L \le \mathrm{q}_\alpha(L)) - \alpha)].$$

Here we take the quantile defined by

$$q_u(L) = \inf\{x | P[L \le x] \ge u\}$$

the smallest u-quantile

Since$ES_\alpha = E[Lg_\alpha(L)]]$ with the function

$$g_\alpha(Y) := (1-\alpha)^{-1}(\mathbf{1}_{\{Y>\mathrm{q}_\alpha(Y)\}} + \beta_Y \mathbf{1}_{\{Y=\mathrm{q}_\alpha(Y)\}}), \tag{3}$$

where β_Y is a real number and

$$\beta_Y := \frac{\mathbb{P}(Y \leq \mathrm{q}_\alpha(Y)) - \alpha}{\mathbb{P}(Y = \mathrm{q}_\alpha(Y))} \quad \text{if} \quad \mathbb{P}(Y = \mathrm{q}_\alpha(Y)) > 0.$$

the density of the associated maximal scenario turns out to be the function g_α. [1]

[1]Note that $\mathrm{ES}_\alpha(Y) = \mathrm{E}(Y \cdot g(Y))$ and $\mathrm{ES}_\alpha(X) \geq \mathrm{E}(X \cdot g(Y))$ for every $X, Y \in V$.

Risk aversion function

It is well-known that

$$ES_\alpha = (1-\alpha)^{-1} \int_\alpha^1 q_u(L)du.$$

Therefore (Acerbi (2002)) the risk aversion weight function formulating risk aversion weights to quantiles of the loss distribution associated with ES_α can be viewed as

$$w_{ES_\alpha}(u) \;=\; (1-\alpha)^{-1}\mathbf{1}_{\{u>\alpha\}}. \tag{4}$$

General risk aversion:

Definition 2 *Let w be an increasing function from $[0,1]$ such that $\int_0^1 w(u)du = 1$, then the map r_w defined by*

$$r_w(L) = \int_0^1 w(u) q_u(L) du$$

is called a spectral risk measure with weight function w.

The name spectral risk measure comes from the representation

$$r_w(X) = \int_0^1 ES_\alpha (1-\alpha) \mu_u(da) \tag{5}$$

$$\text{with the spectral measure } \mu((0,b]) = w(b). \tag{6}$$

Density of spectral risk measures

Theorem 1 *Let r_w be the spectral risk measures with weight function w. The density of the scenario associated with the risk measure equals*

$$L_w \ := \ g_w(L) := \int_0^1 g_\alpha(L)(1-\alpha)\mu(d\alpha) \tag{7}$$

and the generalized scenarios are

$$\mathcal{Q}_w = \{Q | dQ/dP = g_w(Y), Y \in L_\infty\}.$$

*Here $g_\alpha(L)$ is defined in formula (**??**). In particular*

$$r_w(L) \ = \ E[LL_w] \tag{8}$$

Proof: We have

$$r_w(L) \ = \ \int_0^1 ES_\alpha(L)(1-\alpha)\mu(d\alpha)$$

$$
\begin{aligned}
&= \int_0^1 E[Lg_\alpha(L)](1-\alpha)\mu(d\alpha) \ (= E[Lg_w(L)]) \\
&= \int_0^1 \max\{E[Lg_\alpha(Y)]|Y \in L_\infty\}\}(1-\alpha)\mu(d\alpha) \\
&\geq \max\{E[L\int_0^1 g_\alpha(Y)(1-\alpha)\mu(d\alpha)]|Y \in L_\infty\} \\
&= \max\{E[Lg_w(Y)]|\forall Y \in L_\infty\} \\
&\geq E[Lg_w(L)]
\end{aligned}
$$

Hence

$$
r_w(L) = \max\{E[Lg_w(Y)]|\forall Y \in L_\infty\} = E[Lg_w(L)].
$$

$\diamond$

Axiomatic Capital Allocation

Motivation

- Total cost of risk is known, in terms of a monetary risk measure
- How much is caused by a division or even a single transaction (”Causal Capital Allocation”), cf. Expected Shortfall $E[L_i|L > q_\alpha(L)]$.
- Allocation rule in order to improve the entire risk/return profile, cf. Tasche, Littermann, ..., Allocation=sensitivity
- How much should the bank allocate to a division or a single transaction from an axiomatic point of view, cf. Kalkbrener, Kalkbrener,Lotter & Overbeck (RISK 2004).

Axioms

Capital allocation Λ is a real-valued function on $K \bigotimes K$ where K is a space of random variable (Interpretation $\lambda(Y, Z)$ is the capital allocated to Y viewed as a subportfolio of Z)

Definition 3 The capital allocation Λ is called

linear :	$\Lambda(aX + bY, Z) = a\Lambda(X, Z) + b\Lambda(Y, Z)$	$\forall a, b \in \mathbb{R}, X, Y, Z \in V,$
diversifying:	$\Lambda(X, Y) \leq \Lambda(X, X)$	$\forall X, Y \in V,$
continuous at Y:	$\lim_{\epsilon \to 0} \Lambda(X, Y + \epsilon X) = \Lambda(X, Y)$	$\forall X \in V.$

The real valued function ρ on V, defined by

$$\rho(X) = \Lambda(X, X),$$

is called the risk measure associated with Λ.

Asociated Risk Measure

It turns out that a linear and diversifying capital allocation, which is continuous at a portfolio $Y \in V$, is uniquely determined by its associated risk measure, i.e. the diagonal values of Λ.

Theorem 2 *Let Λ be a linear, diversifying capital allocation. If Λ is continuous at $Y \in V$ then for all $X \in V$*

$$\Lambda(X, Y) = \lim_{\epsilon \to 0} \frac{\rho(Y + \epsilon X) - \rho(Y)}{\epsilon}.$$

This theorem shows that the three axioms are sufficient to uniquely determine a capital allocation scheme.

Coherency

For a given risk measure ρ consider the following subset

$$H_\rho := \{h \in V^* \mid h(X) \leq \rho(X) \text{ for all } X \in V\}.$$

Provided that ρ is positively homogeneous and sub-additive then for every $Y \in V$ there exists an $h_Y^\rho \in H_\rho$ with $h_Y^\rho(Y) = \rho(Y)$. This allows to define a capital allocation Λ_ρ by

$$\Lambda_\rho(X, Y) := h_Y^\rho(X). \tag{9}$$

The following theorem states the equivalence between positively homogeneous, sub-additive risk measures and linear, diversifying capital allocations.

Theorem 3 **(a)** *If there exists a linear, diversifying capital allocation Λ with associated risk measure ρ then ρ is positively homogeneous and sub-additive.*
(b) *If ρ is positively homogeneous and sub-additive then Λ_ρ is a linear, diversifying capital allocation with associated risk measure ρ.*

Spectral Capital Allocation

For a spectral risk measures with representing density we have that $h_L(X) = E[g_w(L)X]$ and this coincides with the spectral capital allocation:

$$\Lambda_w(X, L) = E[X g_w(L)]. \tag{10}$$

Intuitively, the capital allocated to transaction or subportfolio X in a portfolio L equals its expectation under the generalized maximal scenario associated with w.

Examples

1. As a first step in the application of spectral risk measures one might think to give to different loss probability levels different weight. This is a straightforward extension of expected shortfall. One might view Expected Shortfall at the 99%-level view as a risk aversion which ignores losses below the 99%-quantile and all losses above the 99%-quantile have the same influence. From an investors point of view this means that only senior debts are cushioned by risk capital. One might on the other hand also be aware of losses which occur more frequently, but of course with a lower aversion than those appearing rarely.

   As a concrete example one might set that losses up to the 50% confidence level should have zero weights, losses between 50% and 99% should have a weight w_0 and losses above the 99%-quantile should have a weight of $k_1 w_0$ and above the 99.9% quantile it should have a weight of $k_2 w_0$. The first tranch from 50% to 99% correspond to an investor in junior debt, and the tranch from 99% to 99.9%

to a senior investor and above the 99.9% a super senior investor or the regulators are concerned. This gives a step function for w:

$$w(u) \;=\; w_0 \mathbf{1}_{\{0.99>u>0.5\}} + k_1 w_0 \mathbf{1}_{\{0.999>u>0.99\}} + k_2 w_0 \mathbf{1}_{\{1>u>0.999\}}$$

The parameter w_0 should be chosen such that the integral over w is still 1.

2. A more continuous form of this is an exponential function starting at a point u_0 between 0 and 1 and then increasing up to 1

$$w(u) = \mathbf{1}_{\{u>u_0\}} \exp(\kappa u)$$

with some constant κ.

Remarks:

1. Expected Shortfall Allocation which allocates the average loss of transaction i in all cases where the overall portfolio capital exceeds a certain quantile can be interpreted as a causal capital allocation. Literally the actual contribution of the transaction to the overall capital is allocated if the conditional expectation is used. In the same way the spectral allocation - at least when the weight function is a step function - is a causal allocation. Here of course the future loss where the capital is needed for get a different weight than those obtained by simple conditional expectation as in Expected Shortfall contribution.

2. Also from the point of view that all actual losses have different impact or subsequent losses. A large loss which is reported in the press might have consequent losses - due to reputational impacts - exceeding the first actual loss by far, and might even damage the capital basis. On the other hand small losses are directly covered by income and will effect the capital not at all. Therefore a different weighting of different loss sizes might be useful.

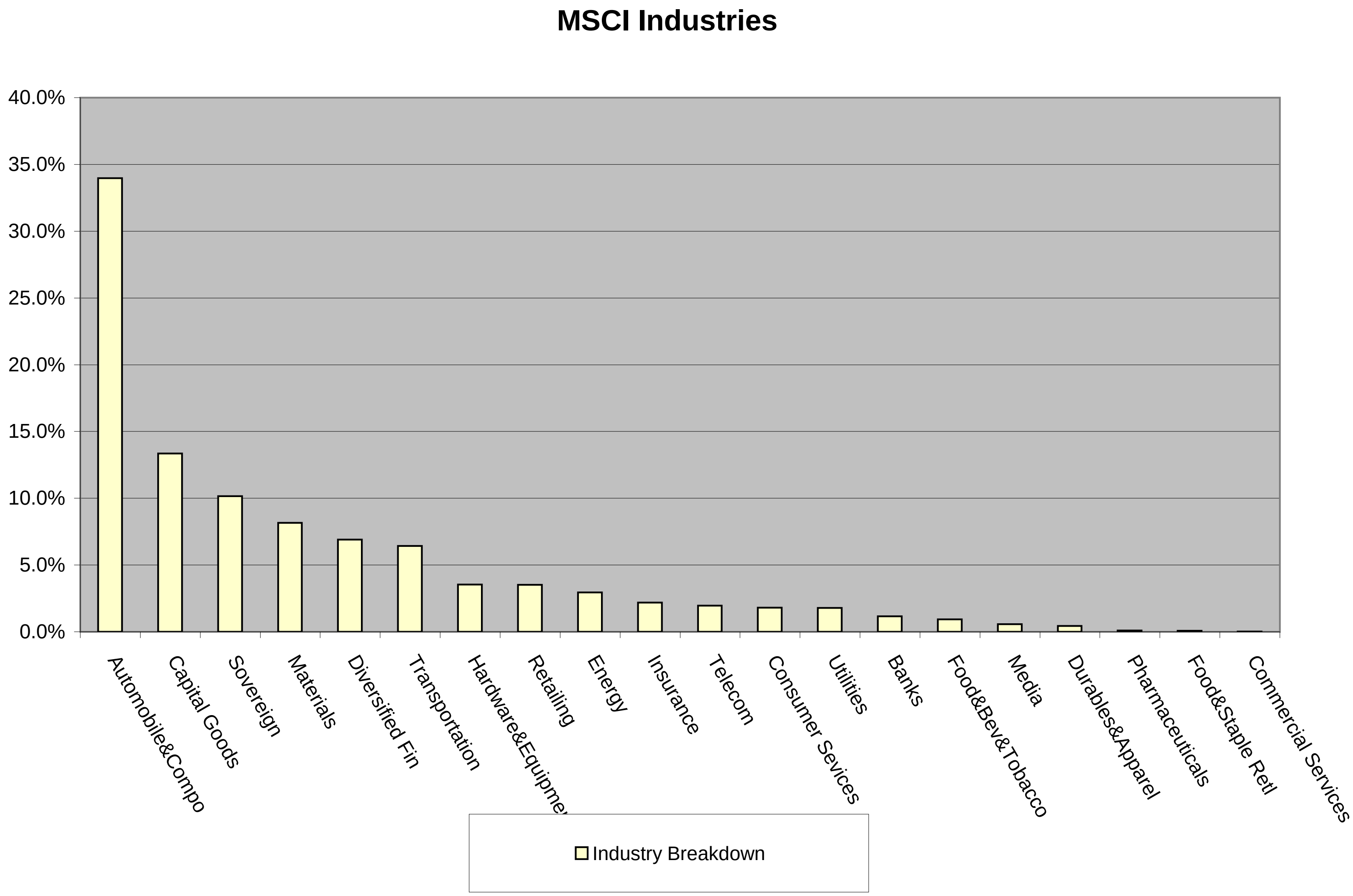
3. In the case of a continuous distribution one can rewrite

$$\int_0^1 w(u)q_u(L)du = E[w(U)q_U(L)] = E[w(F(L))L].$$

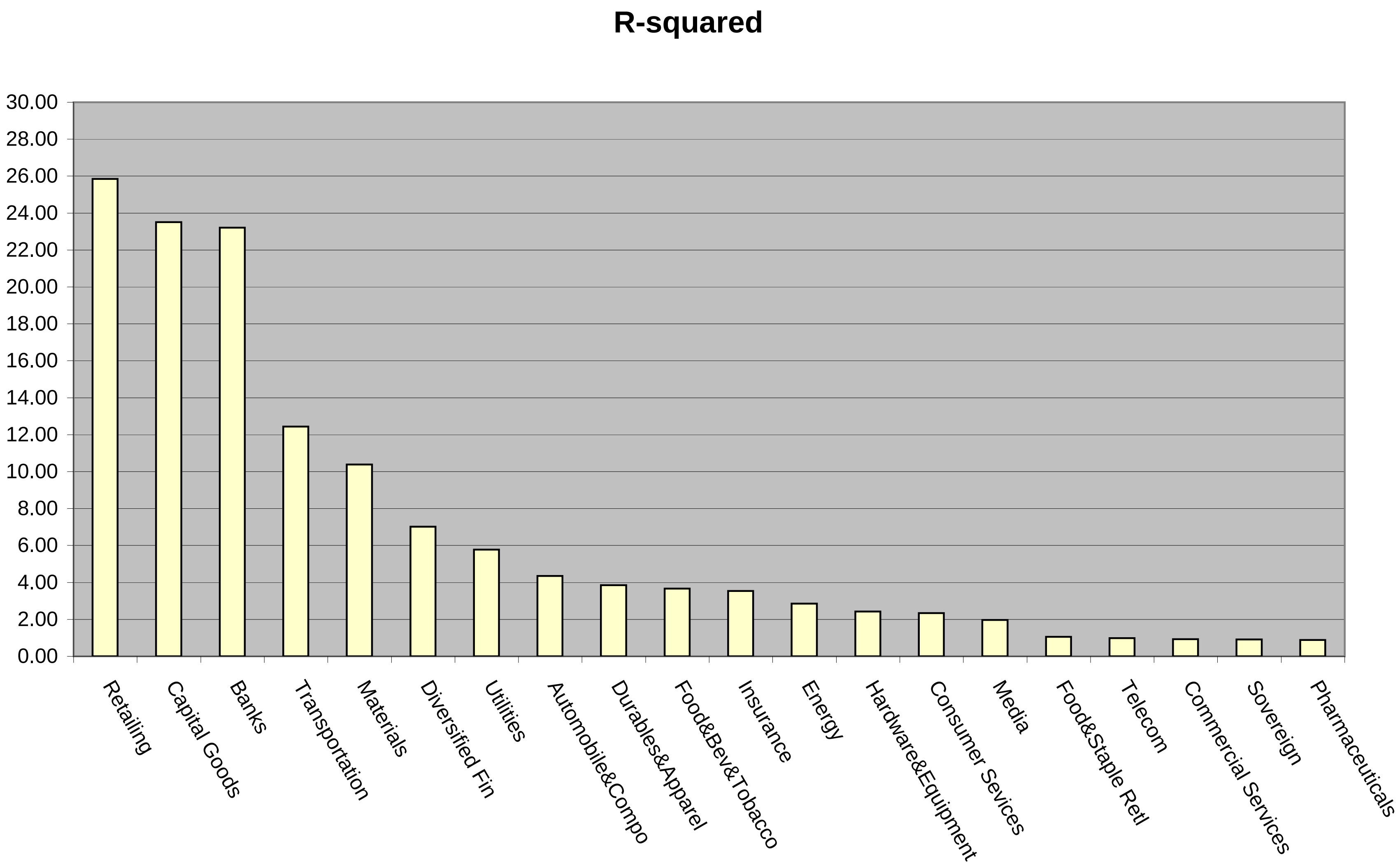
Then the calibration of the weight function can be done in terms of portfolio loss itself instead of the quantiles of the loss distribution. However the new weight function, now defined on the range of the loss variable L, has to be transformed $w_F(x) := w(F(x))$

Example

The portfolio consists of 279 assets with total notional EUR 13.7bn and the following industry breakdown and the portfolio correlation structure is obtained from the R^2 and the correlation structure of the industry and regional factors. The R^2 is the R^2 of the one-dimensional regression of the asset returns with respect to its composite factor, modeled as the sum of industry and country factor. The underlying factor model is based on 24 MSCI Industries and 7 MSCI Regions. The weighted average R^2 is 0.5327.



MSCI Industries
40.0%
35.0%
30.0%
25.0%
20.0%
15.0%
10.0%
5.0%
0.0%
Automobile&Compo
Capital Goods
Sovereign
Materials
Diversified Fin
Transportation
Retailing
Energy
Insurance
Telecom
Consumer Sevices
Utilities
Banks
Food&Bev&Tobacco
Media
Durables&Apparel
Pharmaceuticals
Food&Staple Retl
Commercial Services
Industry Breakdown


R-squared
30.00
28.00
26.00
24.00
22.00
20.00
18.00
16.00
14.00
12.00
10.00
8.00
6.00
4.00
2.00
0.00
Retailing
Capital Goods
Banks
Transportation
Materials
Diversified Fin
Utilities
Automobile&Compo
Durables&Apparel
Food&Bev&Tobacco
Insurance
Energy
Hardware&Equipment
Consumer Sevices
Media
Food&Staple Retl
Telecom
Commercial Services
Sovereign
Pharmaceuticals

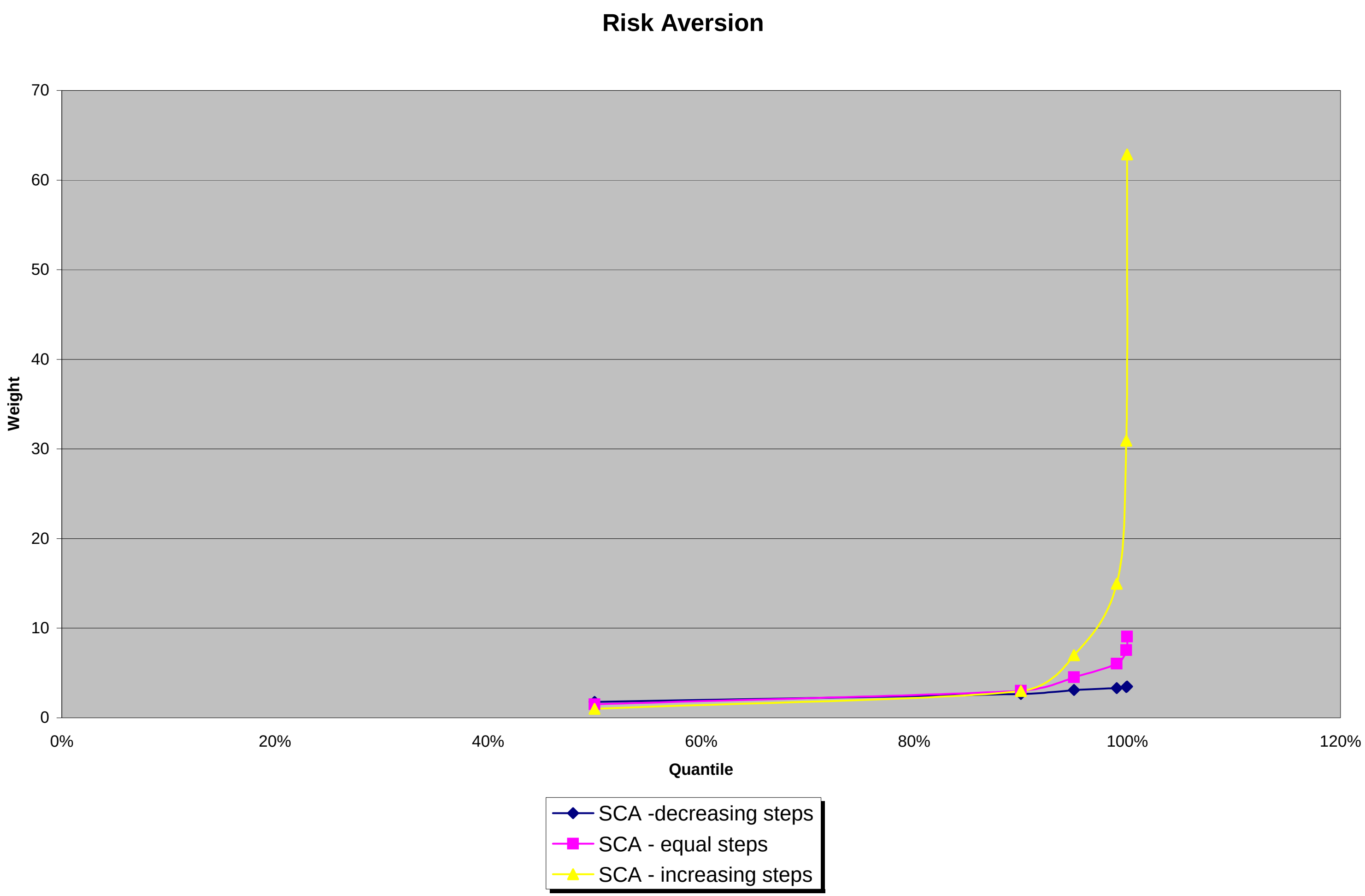
The risk contributions are calculated at quantiles $50\%, 90\%, 95\%, 99\%, 99.9\%$ and 99.98%.

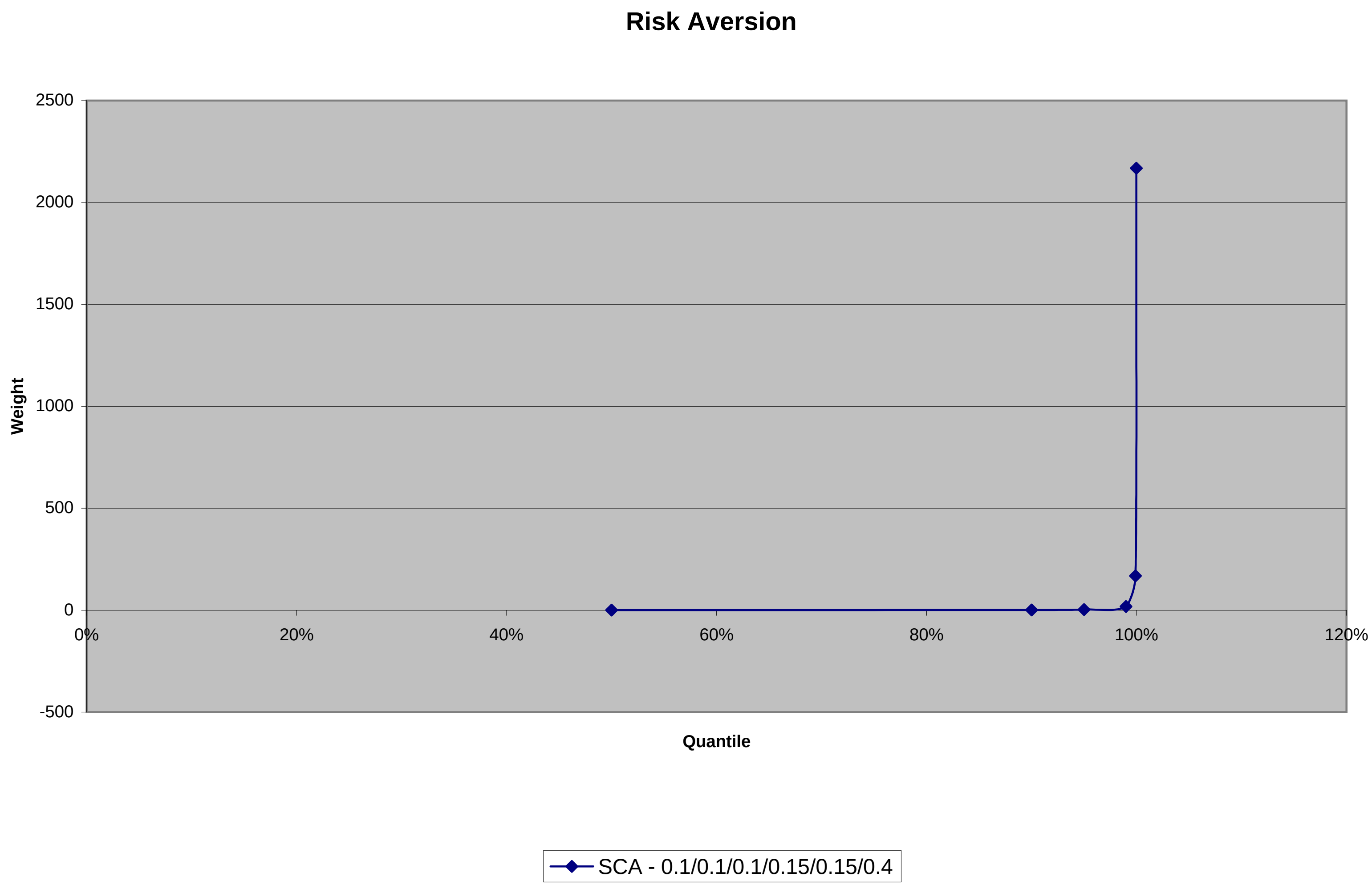
The charts below shows the total Expected Shortfall Contributions allocated to the industries normalized with respect to automobile industry risk contributions and ordered by $\text{ESC}_{99\%}$.

In order to capture all risks of the portfolio a risk measure, which combines few quantile levels, is needed.

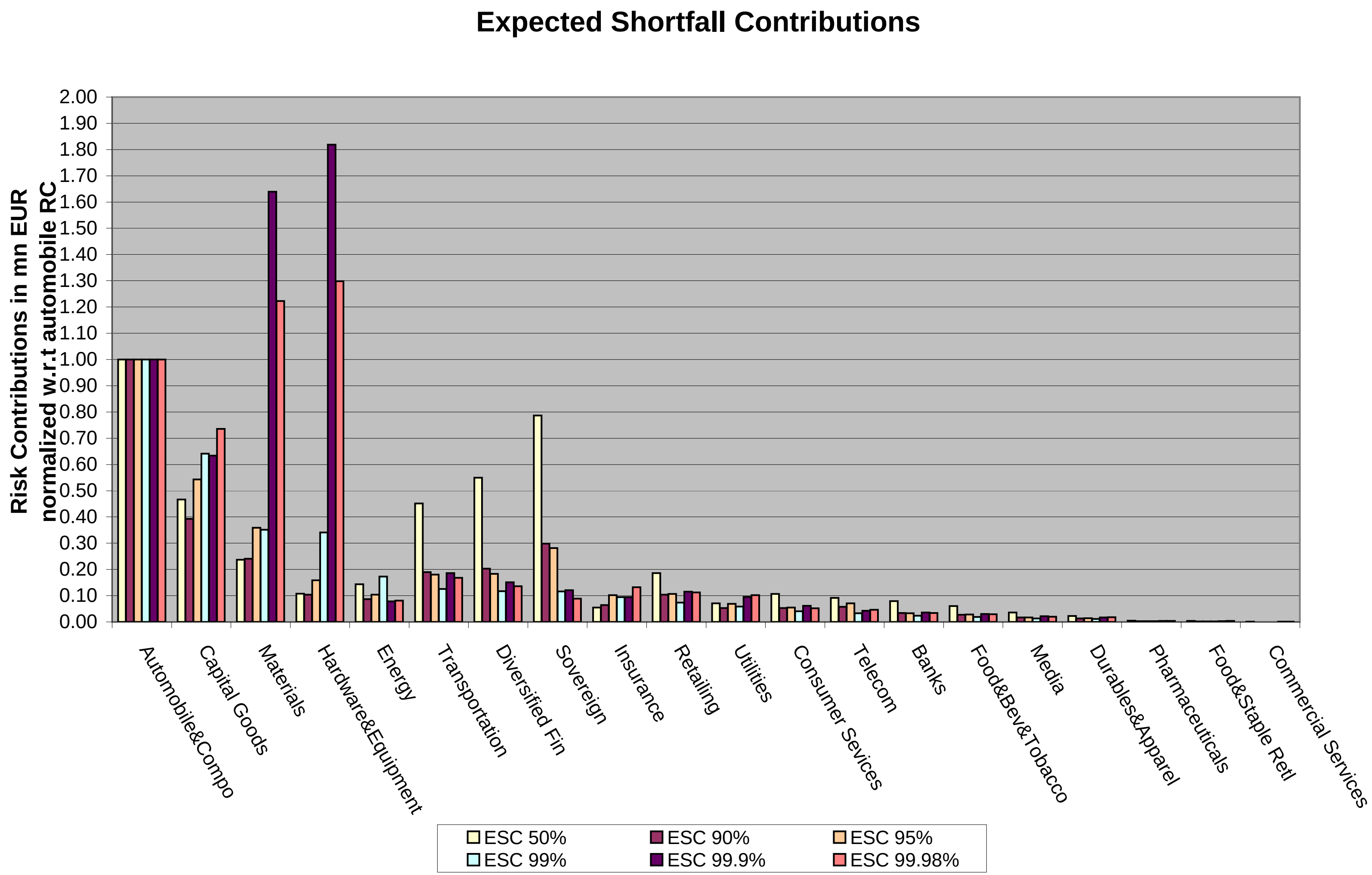
The spectral risk measure as a convex combination of Expected Shortfall risk measures at the following quantiles $50\%, 90\%, 95\%, 99\%, 99.9\%$ and 99.98% can capture both effects, at the tail and at the median of the loss distribution.

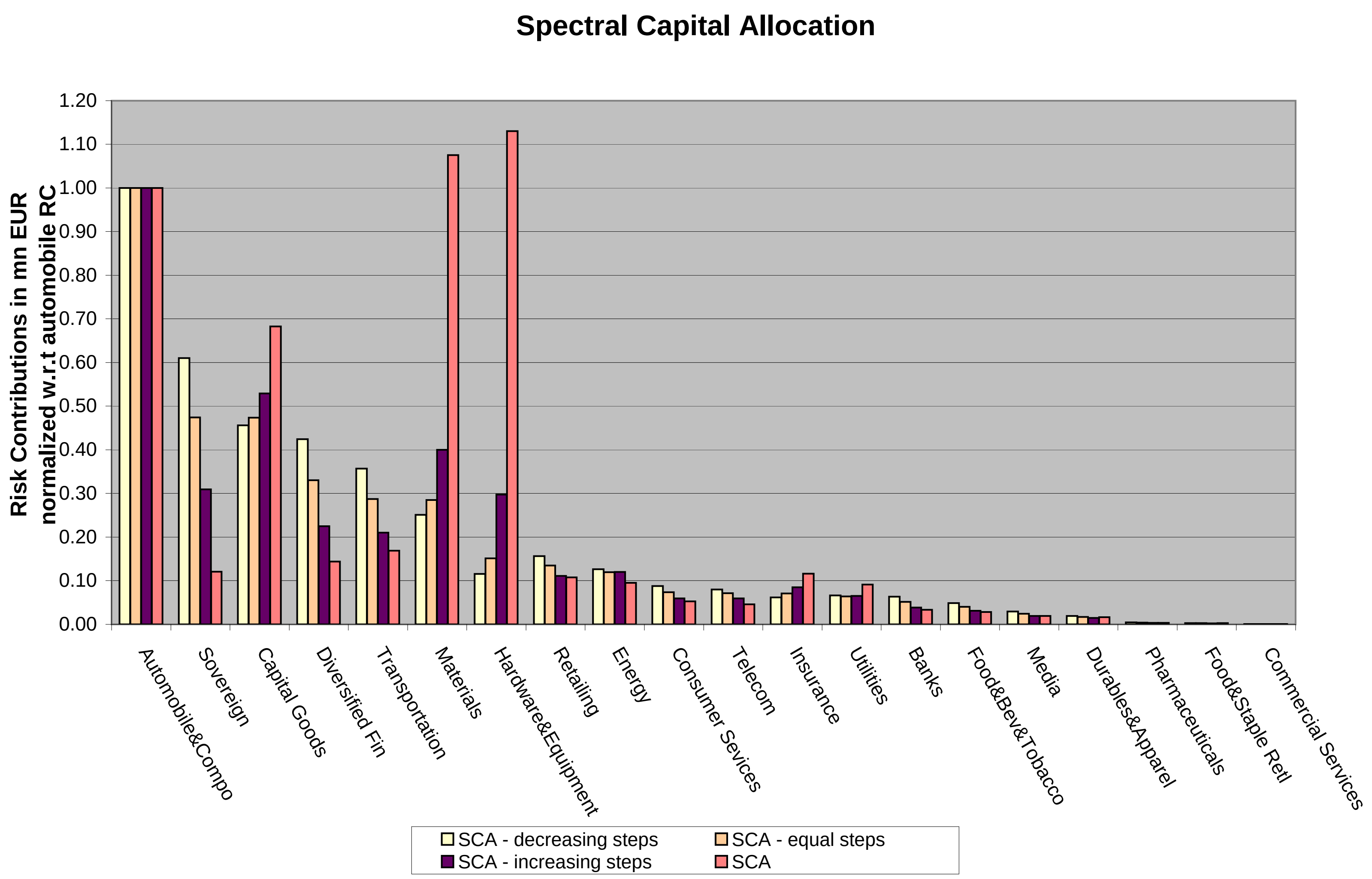
Four spectral risk measures are calculated. The first three are calibrated in terms of increase of the risk aversion function at each considered quantile. The least conservative one is ”SCA - decreasing steps” in which the risk aversion increases at each quantile by half the size it has increased at the quantile before. ”SCA - equal steps” increases in risk aversion by the same amount at each quantile, ”SCA - increasing steps” increases in risk aversion at each quantile by doubling the increase at each quantile. The last most conservative one is SCA - 0.1/0.1/0.1/0.15/0.15/0.4, in which the weights of μ are directly set to 0.1 at the 50%, 90%, 95%- quantiles, 0.15 at the 99% and 99.9%- quantiles and 0.4 at the 99.98%-quantile. The last one has a very steep increase in the risk aversion at the extreme quantiles.


Risk Aversion
Weight
Quantile
0
10
20
30
40
50
60
70
0%
20%
40%
60%
80%
100%
120%
SCA -decreasing steps
SCA - equal steps
SCA - increasing steps



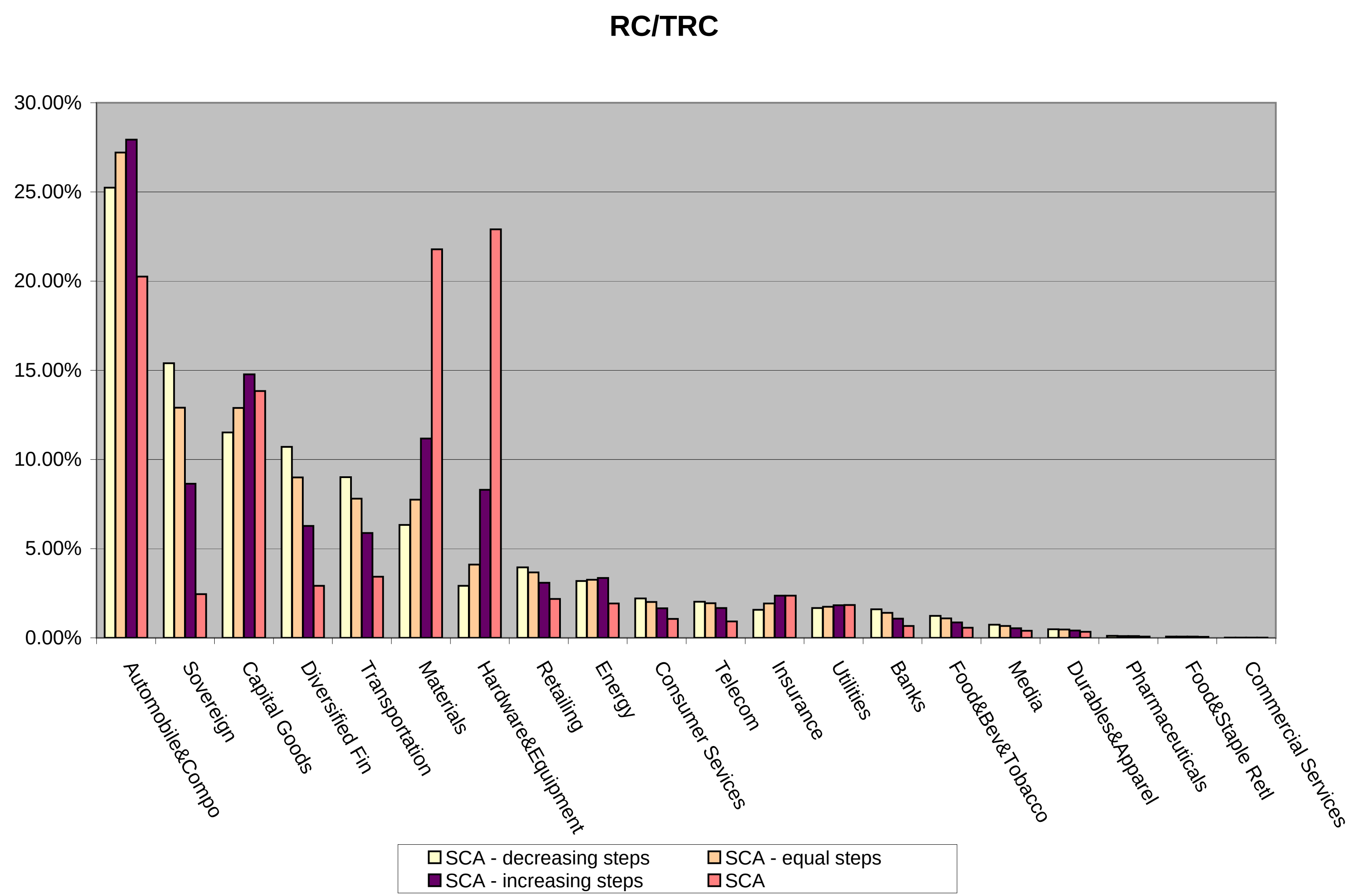
Risk Aversion
2500
2000
1500
1000
500
0
-500
Weight
0%
20%
40%
60%
80%
100%
120%
Quantile
SCA - 0.1/0.1/0.1/0.15/0.15/0.4


Expected Shortfall Contributions
Risk Contributions in mn EUR normalized w.r.t automobile RC
2.00
1.90
1.80
1.70
1.60
1.50
1.40
1.30
1.20
1.10
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
Automobile&Compo
Capital Goods
Materials
Hardware&Equipment
Energy
Transportation
Diversified Fin
Sovereign
Insurance
Retailing
Utilities
Consumer Sevices
Telecom
Banks
Food&Bev&Tobacco
Media
Durables&Apparel
Pharmaceuticals
Food&Staple Retl
Commercial Services
ESC 50%
ESC 90%
ESC 95%
ESC 99%
ESC 99.9%
ESC 99.98%


Spectral Capital Allocation
Risk Contributions in mn EUR normalized w.r.t automobile RC
1.20
1.10
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
Automobile&Compo
Sovereign
Capital Goods
Diversified Fin
Transportation
Materials
Hardware&Equipment
Retailing
Energy
Consumer Sevices
Telecom
Insurance
Utilities
Banks
Food&Bev&Tobacco
Media
Durables&Apparel
Pharmaceuticals
Food&Staple Retl
Commercial Services
SCA - decreasing steps
SCA - equal steps
SCA - increasing steps
SCA

Usually one uses as well percentage figures and risk return figures for portfolio management. On the chart ”RC/TRC” the percentage of total risk (TRC) allocated to the specific industries is displayed.

For the risk management the next table showing allocated risk capital per exposure is very useful. It compares the riskiness of the industries normalized by their exposure. Intuitively it means that if you increase the exposure in ”transportation” by a small amount like 100.000 Euro than the additionally capital measured by SCA-increasing steps will increase by 2.5%, i.e. by 2.5000 Euro. In that sense it gives the marginal capital rate in each industry class. Here the sovereign class is the most risky one. In that portfolio the sovereign exposure was a single transaction with a low rated country and it is therefore no surprise that ”sovereign” performance worst in all risk measures.


RC/TRC
30.00%
25.00%
20.00%
15.00%
10.00%
5.00%
0.00%
Automobile&Compo
Sovereign
Capital Goods
Diversified Fin
Transportation
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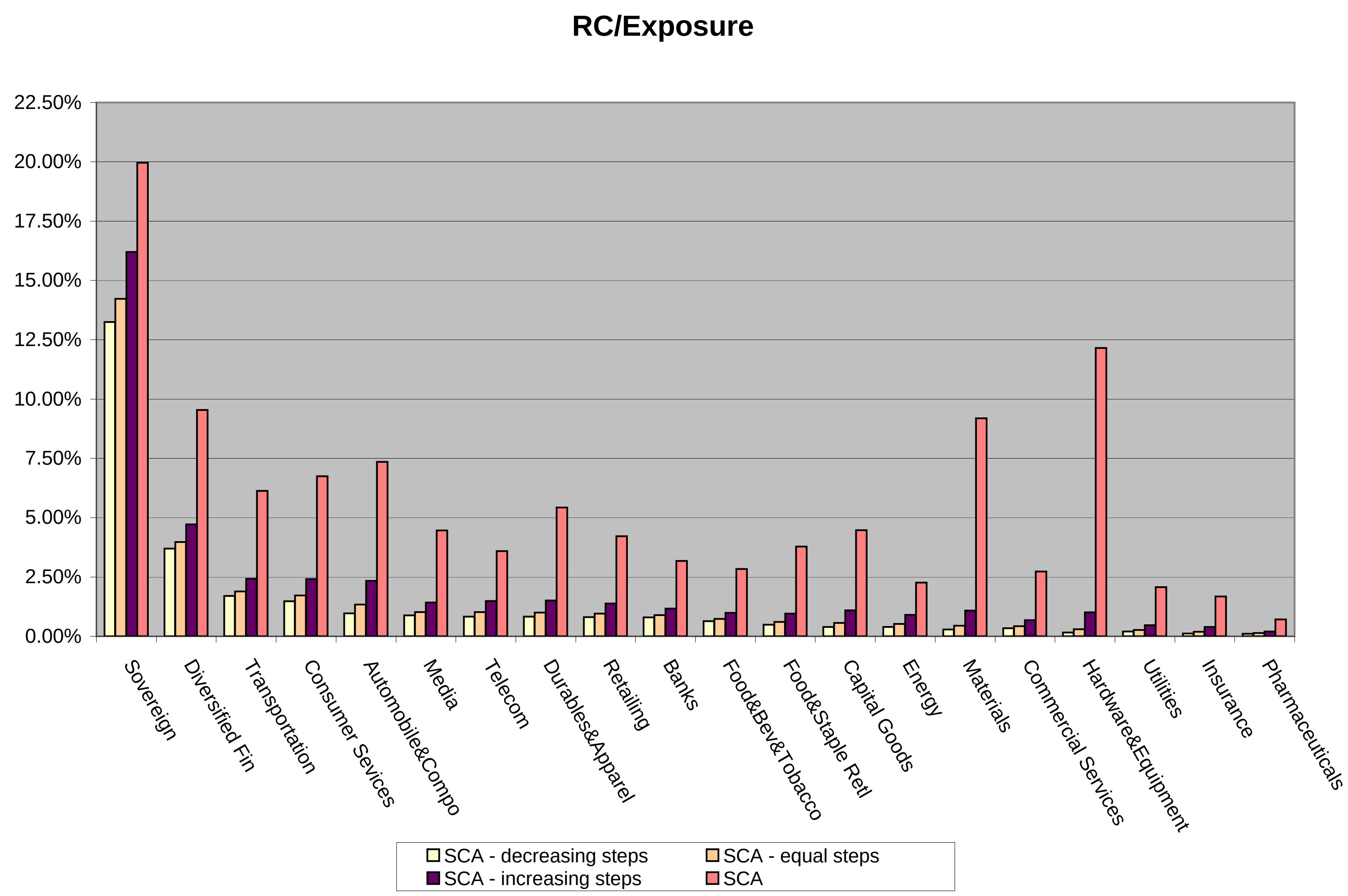
Consumer Sevices
Telecom
Insurance
Utilities
Banks
Food&Bev&Tobacco
Media
Durables&Apparel
Pharmaceuticals
Food&Staple Retl
Commercial Services
SCA - decreasing steps
SCA - equal steps
SCA - increasing steps
SCA


RC/Exposure
22.50%
20.00%
17.50%
15.00%
12.50%
10.00%
7.50%
5.00%
2.50%
0.00%
Sovereign
Diversified Fin
Transportation
Consumer Sevices
Automobile&Compo
Media
Telecom
Durables&Apparel
Retailing
Banks
Food&Bev&Tobacco
Food&Staple Retl
Capital Goods
Energy
Materials
Commercial Services
Hardware&Equipment
Utilities
Insurance
Pharmaceuticals
SCA - decreasing steps
SCA - equal steps
SCA - increasing steps
SCA

With that information one should now be in the position to judge about the possible choice of the most sensible spectral risk measure among the four presented. The measure denoted by SCA based on the weights 0.1,0.1,0.1, 0.15,0.15, 0.4, overemphasis tail risk and ignores volatility risk like the 50%-quantile. From the other three spectral risk measures, also the risk aversion function of the one with increasing steps, does emphasis too much the higher quantiles. SCA decreasing steps seems to punished counterparties with a low rating very much, it seems to a large extend expected loss driven, which can be also seen in the following table on the RAROC-type figures EL/SCA. On that table ”decreasing steps” does not show much dispersion. One could in summary therefore recommend SCA-equal steps.