Wiener-Hopf factorization and distribution of extrema for a family of Lévy processes

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1. Introduction
   - Wiener-Hopf factorization
   - Well-known examples
2. β-family of Lévy processes
3. Distribution of extrema
4. Exit problem for an interval
5. Numerical examples

Review of the Wiener-Hopf factorization

The characteristic exponent $\Psi(z)$ is defined as

$$\mathbb{E}\left[e^{\mathrm{i}zX_t}\right] = \exp(-t\Psi(z)),$$

The Lévy-Khintchine representation for $\Psi(z)$:

$$\Psi(z) = \frac{\sigma^2 z^2}{2} - \mathrm{i}\mu z - \int_{\mathbb{R}} \left(e^{\mathrm{i}zx} - 1 - \mathrm{i}zx\mathbb{I}(|x|<1)\right)\Pi(\mathrm{d}x)$$

We define the extrema processes $\overline{X}_t = \sup\{X_s : s \le t\}$ and $\underline{X}_t = \inf\{X_s : s \le t\}$, introduce an exponential random variable $\mathrm{e}(q)$ with parameter $q > 0$, which is independent of the process X_t, and use the following notation for the characteristic functions of $\overline{X}_{\mathrm{e}(q)}$, $\underline{X}_{\mathrm{e}(q)}$:

$$\phi_q^+(z) = \mathbb{E}\left[e^{\mathrm{i}z\overline{X}_{\mathrm{e}(q)}}\right], \quad \phi_q^-(z) = \mathbb{E}\left[e^{\mathrm{i}z\underline{X}_{\mathrm{e}(q)}}\right]$$

Review of the Wiener-Hopf factorization

Theorem

- *Random variables* $\overline{X}_{e(q)}$ *and* $X_{e(q)} - \overline{X}_{e(q)}$ *are independent.*
- $X_{e(q)} - \overline{X}_{e(q)} \stackrel{d}{=} \underline{X}_{e(q)}$.
- *Random variable* $\overline{X}_{e(q)}$ *[*$\underline{X}_{e(q)}$*] is infinitely divisible, positive [negative] and has zero drift.*

For $z \in \mathbb{R}$ *we have*

$$\frac{q}{q+\Psi(z)} = \phi_q^+(z)\phi_q^-(z).$$

Outline

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WH for Brownian motion with drift

The main idea: since the random variable $\overline{X}_{e(q)}$ $[\underline{X}_{e(q)}]$ is positive [negative], its characteristic function must be analytic and have no zeros in $\mathbb{C}^+$ $[\mathbb{C}^-]$, where

$$\mathbb{C}^+ = \{z \in \mathbb{C} \ : \ \text{Im}(z) > 0\}, \ \ \mathbb{C}^- = \{z \in \mathbb{C} \ : \ \text{Im}(z) < 0\}, \ \ \bar{\mathbb{C}}^{\pm} = \mathbb{C}^{\pm} \cup \mathbb{R}.$$

Example:

Let $X_t = W_t + \mu t$. Then $\Psi(z) = \frac{z^2}{2} - i\mu z$ and the equation $q + \Psi(z) = 0$ has two solutions

$$z_{1,2} = \mathrm{i}(\mu \pm \sqrt{\mu^2 + 2q})$$

WH for Brownian motion with drift

Function $q/(\Psi(z)+q)$ can be factorized as

$$\frac{q}{q+\Psi(z)} = \frac{q}{\frac{z^2}{2}-\mathrm{i}\mu z+q} = \frac{\mu+\sqrt{\mu^2+2q}}{\mathrm{i}z+\mu+\sqrt{\mu^2+2q}} \times \frac{\mu-\sqrt{\mu^2+2q}}{\mathrm{i}z+\mu-\sqrt{\mu^2+2q}}$$

Thus

$$\phi_q^+(z) = \frac{-\mathrm{i}(\mu-\sqrt{\mu^2+2q})}{z-\mathrm{i}(\mu-\sqrt{\mu^2+2q})}$$

and $\overline{X}_{\mathrm{e}(q)}$ is an exponential random variable with parameter $\sqrt{\mu^2+2q}-\mu$.

Kou model: double exponential jump diffusion model

X_t is a Lévy process with jumps defined by

$$\pi(x) = a_1 e^{-b_1 x} \mathbf{I}_{\{x>0\}} + a_2 e^{b_2 x} \mathbf{I}_{\{x<0\}}$$

Then the characteristic exponent is

$$\Psi(z) = \frac{\sigma^2 z^2}{2} - i\mu z - \frac{a_1}{b_1 - \mathrm{i}z} - \frac{a_2}{b_2 + \mathrm{i}z} + \frac{a_1}{b_1} + \frac{a_2}{b_2}$$

Thus equation $q + \Psi(z) = 0$ is a *fourth degree polynomial equation*, and we have explicit solutions and exact WH factorization.

Phase-type distributed jumps

Definition

The distribution of the first passage time of the finite state continuous time Markov chain is called *phase-type* distribution.

$$q(x) = \mathbf{p_0} e^{x\mathcal{L}} \mathbf{e_1}$$

where b_i are eigenvalues of the Markov generator $\mathcal{L}$. Thus if X_t has phase-type jumps, its characteristic exponent $\Psi(z)$ is a *rational* function, and $q + \Psi(z) = 0$ is reduced to a polynomial equation, and the Wiener-Hopf factors are given in closed form (in terms of the roots of this polynomial equation).

Outline

Definition of the β-family

Definition

We define the β-family of Lévy processes by the generating triple (μ, σ, π), where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and the density of the Lévy measure is

$$\pi(x) = c_1 \frac{e^{-\alpha_1 \beta_1 x}}{(1 - e^{-\beta_1 x})^{\lambda_1}} \mathbf{I}_{\{x>0\}} + c_2 \frac{e^{\alpha_2 \beta_2 x}}{(1 - e^{\beta_2 x})^{\lambda_2}} \mathbf{I}_{\{x<0\}}$$

and parameters satisfy $\alpha_i > 0$, $\beta_i > 0$, $c_i \geq 0$ and $\lambda_i \in (0, 3)$.

Lévy processes similar to the β-family

The generalized tempered stable family

$$\pi(x) = c_+ \frac{e^{-\alpha_+ x}}{x^{\lambda_+}} \mathbf{I}_{\{x>0\}} + c_- \frac{e^{\alpha_- x}}{|x|^{\lambda_-}} \mathbf{I}_{\{x<0\}}.$$

can be obtained as the limit as $\beta \to 0^+$ if we let

$$c_1 = c_+ \beta^{\lambda_+}, \quad c_2 = c_- \beta^{\lambda_-}, \quad \alpha_1 = \alpha_+ \beta^{-1}, \quad \alpha_2 = \alpha_- \beta^{-1}, \quad \beta_1 = \beta_2 = \beta$$

Particular cases:

- $\lambda_1 = \lambda_2 \longrightarrow$ tempered stable, or KoBoL processes
- $c_1 = c_2$, $\lambda_1 = \lambda_2$ and $\beta_1 = \beta_2 \longrightarrow$ CGMY processes

Computing the characteristic exponent

Theorem

If $\lambda_i \in (0,3) \setminus \{1,2\}$ *then*

$$\begin{aligned}\Psi(z) &= \frac{\sigma^2 z^2}{2} + \mathrm{i}\rho z + \gamma \\ &- \frac{c_1}{\beta_1} \mathrm{B}\left(\alpha_1 - \frac{\mathrm{i}z}{\beta_1}; 1-\lambda_1\right) - \frac{c_2}{\beta_2} \mathrm{B}\left(\alpha_2 + \frac{\mathrm{i}z}{\beta_2}; 1-\lambda_2\right).\end{aligned}$$

Here $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function.

Properties

(i) The characteristic exponent $\Psi(z)$ is a meromorphic function which has simple poles at points $\{-\mathrm{i}\rho_n, \mathrm{i}\hat{\rho}_n\}_{n\geq 1}$, where

$$\rho_n = \beta_1(\alpha_1 + n - 1), \quad \hat{\rho}_n = \beta_2(\alpha_2 + n - 1).$$

(ii) For $q \geq 0$ function $q + \Psi(z)$ has roots at points $\{-\mathrm{i}\zeta_n, \mathrm{i}\hat{\zeta}_n\}_{n\geq 1}$ where ζ_n and $\hat{\zeta}_n$ are nonnegative real numbers (strictly positive if $q > 0$).

Properties

(iii) The roots and poles of $q+\Psi(\mathrm{i}z)$ satisfy the following interlacing condition

$$... -\rho_2 < -\zeta_2 < -\rho_1 < -\zeta_1 < 0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < \hat{\rho}_2 < ...$$

(iv) The Wiener-Hopf factors are expressed as convergent infinite products,

$$\begin{aligned} \phi_q^+(\mathrm{i}z) &= \mathbb{E}\left[e^{-z\overline{X}_{\mathrm{e}(q)}}\right] = \prod_{n\geq 1} \frac{1+\frac{z}{\rho_n}}{1+\frac{z}{\zeta_n}} \\ \phi_q^-(-\mathrm{i}z) &= \mathbb{E}\left[e^{z\underline{X}_{\mathrm{e}(q)}}\right] = \prod_{n\geq 1} \frac{1+\frac{z}{\hat{\rho}_n}}{1+\frac{z}{\hat{\zeta}_n}}. \end{aligned}$$

Meromorphic Lévy processes

A. Kuznetsov, A.E. Kyprianou and J.C. Pardo (2010)
"Meromorphic Lévy processes and their fluctuation identities."

The density of the Lévy measure is defined as

$$\pi(x) = \mathbb{I}_{\{x>0\}} \sum_{i=1}^{N} a_i e^{-\rho_i x} + \mathbb{I}_{\{x<0\}} \sum_{i=1}^{\hat{N}} \hat{a}_i e^{\hat{\rho}_i x},$$

where all the coefficients are positive and $N \le \infty$, $\hat{N} \le \infty$. In the case $N = \infty$ $\{ \hat{N} = \infty \}$ the series

$$\sum_{i=1}^{\infty} a_i \rho_i^{-3} \quad \left\{ \sum_{i=1}^{\infty} \hat{a}_i \hat{\rho}_i^{-3} \right\}$$

must converge.

Outline

Main analytical tool: partial fraction decomposition

Lemma

Assume that we have two increasing sequences $\rho = \{\rho_n\}_{n\ge1}$ *and* $\zeta = \{\zeta_n\}_{n\ge1}$ *of positive numbers which satisfy the following conditions.*

(i) *Interlacing condition* $\zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \dots$

(ii) *There exists* $\alpha > 1/2$ *and* $\epsilon > 0$ *such that* $\rho_n > \epsilon n^{\alpha}$ *for all integer numbers* n.

Then we have the following partial fraction decompositions

$$\prod_{n\ge1} \frac{1+\frac{z}{\rho_n}}{1+\frac{z}{\zeta_n}} = \mathrm{a}_0(\rho,\zeta) + \sum_{n\ge1} \mathrm{a}_n(\rho,\zeta)\frac{\zeta_n}{\zeta_n+z},$$

$$\prod_{n\ge1} \frac{1+\frac{z}{\zeta_n}}{1+\frac{z}{\rho_n}} = 1 + z\mathrm{b}_0(\zeta,\rho) + \sum_{n\ge1} \mathrm{b}_n(\zeta,\rho)\left[1-\frac{\rho_n}{\rho_n+z}\right],$$

Main analytical tool: partial fraction decomposition

where

$$\mathrm{a}_0(\rho,\zeta) = \lim_{n\to+\infty} \prod_{k=1}^{n} \frac{\zeta_k}{\rho_k}, \quad \mathrm{a}_n(\rho,\zeta) = \left(1-\frac{\zeta_n}{\rho_n}\right) \prod_{\substack{k\ge 1\\ k\ne n}} \frac{1-\frac{\zeta_n}{\rho_k}}{1-\frac{\zeta_n}{\zeta_k}},$$

$$\mathrm{b}_0(\zeta,\rho) = \frac{1}{\zeta_1} \lim_{n\to+\infty} \prod_{k=1}^{n} \frac{\rho_k}{\zeta_{k+1}}, \quad \mathrm{b}_n(\zeta,\rho) = -\left(1-\frac{\rho_n}{\zeta_n}\right) \prod_{\substack{k\ge 1\\ k\ne n}} \frac{1-\frac{\rho_n}{\zeta_k}}{1-\frac{\rho_n}{\rho_k}}.$$

Vector/matrix notation

Everything will depend on the coefficients $\{\mathrm{a}_n(\rho,\zeta), \mathrm{a}_n(\hat\rho,\hat\zeta)\}_{n\ge 0}$ and $\{\mathrm{b}_n(\zeta,\rho), \mathrm{b}_n(\hat\zeta,\hat\rho)\}_{n\ge 0}$. We define for convenience a column vector

$$\bar{\mathrm{a}}(\rho,\zeta) = [\mathrm{a}_0(\rho,\zeta), \mathrm{a}_1(\rho,\zeta), \mathrm{a}_2(\rho,\zeta), ...]^T$$

and similarly for $\mathrm{a}(\hat\rho,\hat\zeta)$, $\mathrm{b}(\zeta,\rho)$ and $\mathrm{b}(\hat\zeta,\hat\rho)$. Next, given a sequence of positive numbers $\zeta = \{\zeta_n\}_{n\ge 1}$, we define the column vector $\bar{\mathrm{v}}(\zeta,x)$ as a vector of distributions

$$\bar{\mathrm{v}}(\zeta,x) = \left[\delta_0(x), \zeta_1 e^{-\zeta_1 x}, \zeta_2 e^{-\zeta_2 x}, \dots\right]^T,$$

where $\delta_0(x)$ is the Dirac delta function at $x=0$.

Distribution of extrema

Corollary

(i) *For* $x \geq 0$

$$\begin{aligned} \mathbb{P}(\overline{X}_{\mathrm{e}(q)} \in \mathrm{d}x) &= \bar{\mathrm{a}}(\rho, \zeta)^T \times \bar{\mathrm{v}}(\zeta, x)\mathrm{d}x \\ \mathbb{P}(-\underline{X}_{\mathrm{e}(q)} \in \mathrm{d}x) &= \bar{\mathrm{a}}(\hat{\rho}, \hat{\zeta})^T \times \bar{\mathrm{v}}(\hat{\zeta}, x)\mathrm{d}x. \end{aligned}$$

(ii) $\mathrm{a}_0(\rho, \zeta)$ (equiv. $\mathrm{a}_0(\hat{\rho}, \hat{\zeta})$) *is nonzero if and only if* 0 *is irregular for* $(0, \infty)$ (equiv. $(-\infty, 0)$).

(iii) $\mathrm{b}_0(\zeta, \rho)$ (equiv. $\mathrm{b}_0(\hat{\zeta}, \hat{\rho})$) *is nonzero if and only if the process* X_t *creeps upwards.* (equiv. downwards)

Distribution of extrema: notation

Expression in vector/matrix form

$$\mathbb{P}(\overline{X}_{\mathrm{e}(q)} \in \mathrm{d}x) = \bar{\mathrm{a}}(\rho, \zeta)^T \times \bar{\mathrm{v}}(\zeta, x)\mathrm{d}x$$

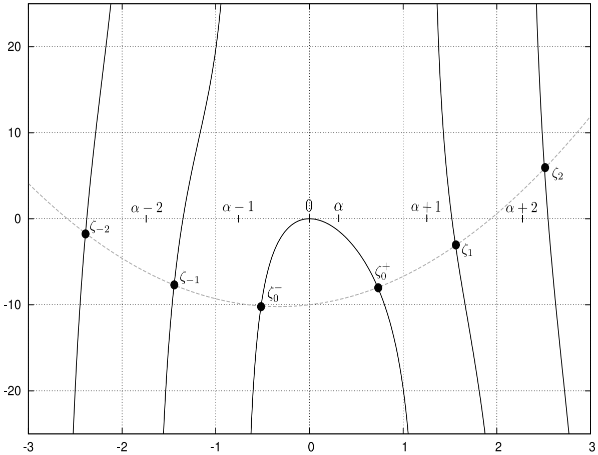
is equivalent to

$$\mathbb{P}(\overline{X}_{\mathrm{e}(q)} = 0) = \mathrm{a}_0(\rho, \zeta)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(\overline{X}_{\mathrm{e}(q)} < x) = \sum_{n \geq 1} \mathrm{a}_n(\rho, \zeta)\zeta_n e^{-\zeta_n x}$$

Computing roots



Joint distribution of the fpt and the overshoot

Define $\tau_a^+ = \inf\{t > 0 : X_t > a\}$.

Theorem

Define a matrix $\mathbf{A} = \{a_{i,j}\}_{i,j\geq 0}$ *as*

$$a_{i,j} = \begin{cases} 0 & \text{if } i = 0,\ j \geq 0 \\ \mathrm{a}_i(\rho,\zeta)\mathrm{b}_0(\zeta,\rho) & \text{if } i \geq 1,\ j = 0 \\ \dfrac{\mathrm{a}_i(\rho,\zeta)\mathrm{b}_j(\zeta,\rho)}{\rho_j - \zeta_i} & \text{if } i \geq 1,\ j \geq 1 \end{cases}$$

Then for $c > 0$ *and* $y \geq 0$ *we have*

$$\mathbb{E}\left[e^{-q\tau_c^+}\mathbb{I}\left(X_{\tau_c^+} - c \in \mathrm{d}y\right)\right] = \bar{\mathrm{v}}(\zeta,c)^T \times \mathbf{A} \times \bar{\mathrm{v}}(\rho,y)\mathrm{d}y.$$

Outline

Two-sided exit problem

Theorem

Let $a > 0$ *and define a matrix* $\mathbf{B} = \mathbf{B}(\hat{\rho}, \zeta, a) = \{b_{i,j}\}_{i,j \geq 0}$ *with*

$$b_{i,j} = \begin{cases} \zeta_j e^{-a\zeta_j} & \text{if } i = 0,\ j \geq 1 \\ 0 & \text{if } i \geq 0,\ j = 0 \\ \dfrac{\hat{\rho}_i \zeta_j}{\hat{\rho}_i + \zeta_j} e^{-a\zeta_j} & \text{if } i \geq 1,\ j \geq 1 \end{cases}$$

and similarly $\hat{\mathbf{B}} = \mathbf{B}(\rho, \hat{\zeta}, a)$*. There exist matrices* $\mathbf{C}_1$*,* $\mathbf{C}_2$ *and* $\hat{\mathbf{C}}_1$*,* $\hat{\mathbf{C}}_2$ *such that for* $x \in (0, a)$ *we have*

$$\mathbb{E}_x \left[e^{-q\tau_a^+} \mathbb{I} \left(X_{\tau_a^+} \in \mathrm{d}y \ ; \ \tau_a^+ < \tau_0^- \right) \right]$$
$$= \left[\bar{\mathrm{v}}(\zeta, a - x)^T \times \mathbf{C}_1 + \bar{\mathrm{v}}(\hat{\zeta}, x)^T \times \mathbf{C}_2 \right] \times \bar{\mathrm{v}}(\rho, y - a) \mathrm{d}y$$

Two-sided exit problem

These matrices satisfy the following system of linear equations

$$\begin{cases} \mathbf{C}_1 &= \mathbf{A} - \hat{\mathbf{C}}_2 \mathbf{B} \mathbf{A} \\ \hat{\mathbf{C}}_2 &= -\mathbf{C}_1 \hat{\mathbf{B}} \hat{\mathbf{A}} \end{cases} \qquad \begin{cases} \hat{\mathbf{C}}_1 &= \hat{\mathbf{A}} - \mathbf{C}_2 \hat{\mathbf{B}} \hat{\mathbf{A}} \\ \mathbf{C}_2 &= -\hat{\mathbf{C}}_1 \mathbf{B} \mathbf{A} \end{cases}$$

This system of linear equations can be solved iteratively with exponential convergence.

Outline

Parameters

We use a process from the β-family with parameters

$$(\sigma, \mu, \alpha_1, \beta_1, \lambda_1, c_1, \alpha_2, \beta_2, \lambda_2, c_2) = (\sigma, \mu, 1, 1.5, 1.5, 1, 1, 1.5, 1.5, 1)$$

Here $\mu = \mathbb{E}[X_1]$ and σ is the Gaussian coefficient, the other parameters define the density of a Lévy measure, which has exponentially decaying tails and $O(|x|^{-3/2})$ singularity at $x = 0$, thus this process has jumps of infinite activity but finite varation. We define the following four parameter sets

Set 1: $\sigma = 0.5, \mu = 1$ Set 2: $\sigma = 0.5, \mu = -1$

Set 3: $\sigma = 0, \mu = 1$ Set 4: $\sigma = 0, \mu = -1$

Double-sided exit problem

(i) density of the overshoot if the exit happens at the upper boundary

$$f_1(x,y) = \frac{\mathrm{d}}{\mathrm{d}y} \mathbb{E}_x \left[e^{-q\tau_1^+} \mathbb{I}\left(X_{\tau_1^+} \le y \; ; \; \tau_1^+ < \tau_0^- \right) \right]$$

(ii) probability of exiting from the interval $[0,1]$ at the upper boundary

$$f_2(x) = \mathbb{E}_x \left[e^{-q\tau_1^+} \mathbb{I}\left(\tau_1^+ < \tau_0^- \right) \right]$$

(iii) probability of exiting the interval $[0,1]$ by creeping across the upper boundary

$$f_3(x) = \mathbb{E}_x \left[e^{-q\tau_1^+} \mathbb{I}\left(X_{\tau_1^+} = 1 \; ; \; \tau_1^+ < \tau_0^- \right) \right]$$

Details of the alrgorithm

- Truncate coefficients $\mathrm{a}_i(\rho,\zeta)$ and $\mathrm{a}_i(\hat{\rho},\hat{\zeta})$ at $i=200$; coefficients $\mathrm{b}_j(\zeta,\rho)$ and $\mathrm{b}_j(\hat{\zeta},\hat{\rho})$ at $j=100$.
- In order to compute coefficients $\mathrm{a}_i(\rho,\zeta)$, $\mathrm{a}_i(\hat{\rho},\hat{\zeta})$, $\mathrm{b}_j(\zeta,\rho)$ and $\mathrm{b}_j(\hat{\zeta},\hat{\rho})$ we truncate the corresponding infinite products at $k=400$
- All the computations depend on precomputing $\{\zeta_n,\hat{\zeta}_n\}$ for $n=1,2,..,400$ (solving $q+\Psi(iz)=0$).
- The code was written in Fortran and the computations were performed on a standard laptop (Intel Core 2 Duo 2.5 GHz processor and 3 GB of RAM).
- Time to produce the three graphs for each parameter set: 0.15 sec.

Double sided exit: $\sigma > 0$ and positive drift

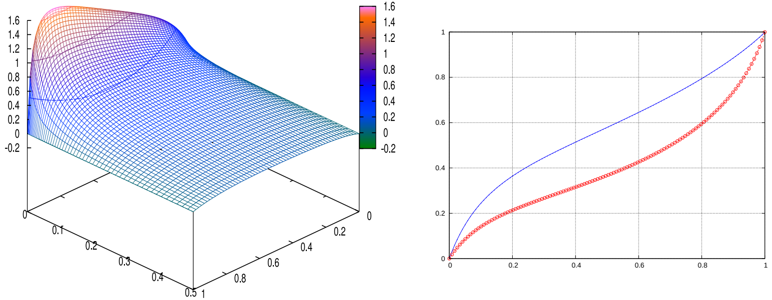


Figure: Unbounded variation case ($\sigma = 0.5$): computing the density of the overshoot $f_1(x,y)$ ($x \in (0,1)$, $y \in (0,0.5)$), probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$ for parameter Set 1, positive drift $\mu = 1$

Double sided exit: $\sigma > 0$ and negative drift

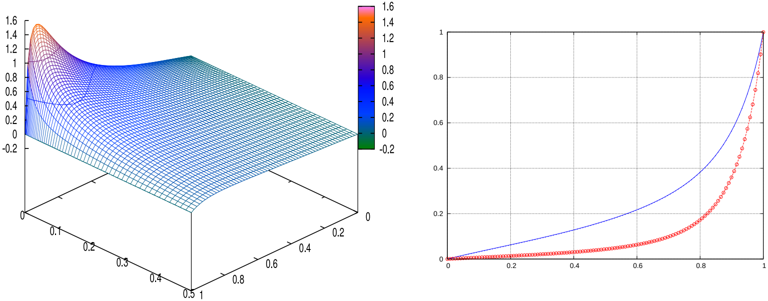


Figure: Unbounded variation case ($\sigma = 0.5$): computing the density of the overshoot $f_1(x, y)$ ($x \in (0, 1)$, $y \in (0, 0.5)$), probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$ for parameter Set 2, negative drift $\mu = -1$.

Double sided exit: bounded variation and positive drift

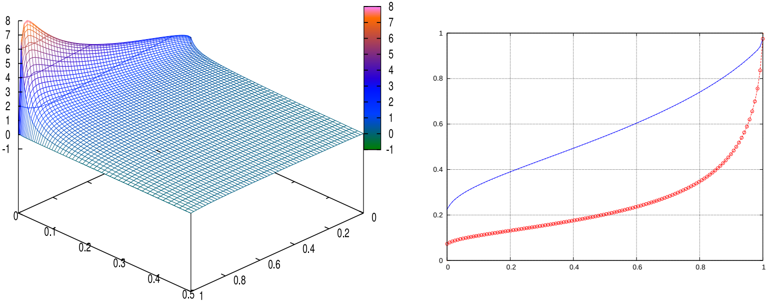


Figure: Bounded variation case ($\sigma = 0$): computing the density of the overshoot $f_1(x, y)$ ($x \in (0, 1)$, $y \in (0, 0.5)$), probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$ for parameter Set 3, positive drift $\mu = 1$.

Double sided exit: bounded variation and negative drift

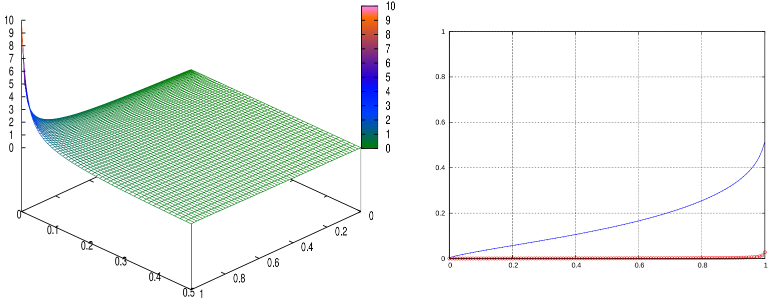


Figure: Bounded variation case ($\sigma = 0$): computing the density of the overshoot $f_1(x, y)$ ($x \in (0, 1)$, $y \in (0, 0.5)$), probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$ for parameter Set 4, positive drift $\mu = -1$.

Time changed Lévy processes

Price of the rebate barrier option with the exponential maturity

$$\pi_X(x,q) = \mathbb{E}_x\left[\mathbb{I}(\tau_a^+ < \mathrm{e}(q)) f(X_{\tau_a^+})\right]$$

Define a time-changed process $Y_s = X_{T_s}$, $s \geq 0$, where we assume that T_s is continuous and independent of X_t. Define s_a^+ to be the first passage time of process Y_s above a. Then the price of the option with the deterministic maturity u is given by

$$\pi_Y(y,u) = \mathbb{E}_y\left[\mathbb{I}(s_a^+ < u) f(Y_{s_a^+})\right] = \frac{1}{2\pi \mathrm{i}} \int\limits_{q_0 + \mathrm{i}\mathbb{R}} \pi_X(y,q) \mathbb{E}\left[e^{qT_u}\right] q^{-2} \mathrm{d}q$$

References:

- A. Kuznetsov (2009)
  "Wiener-Hopf factorization and distribution of extrema for a family of Lévy processes." *to appear in Ann. Appl. Probab.*
- A. Kuznetsov (2009)
  "Wiener-Hopf factorization for a family of Lévy processes related to theta functions." *preprint*
- A. Kuznetsov, A.E. Kyprianou and J.C. Pardo (2010)
  "Meromorphic Lévy processes and their fluctuation identities." *preprint*
- A. Kuznetsov, A.E. Kyprianou, J.C. Pardo, and K. van Schaik (2010)
  "A Wiener-Hopf Monte Carlo simulation technique for Lévy process." *preprint*

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