Real Options and Free-Boundary Problem: A Variational View

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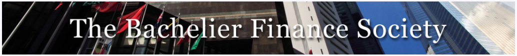

Real Options

Let us consider an investment project, which is characterized by a pair $(\{V_t, t \geq 0\},\ I)$,
where V_t is the Present Value of the project started at time t, and I is the cost of required investment.

V_t is assumed to be a stochastic process, defined at a probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbf{P})$.

The real options model (starting from McDonald-Siegel model) supposes that:

- at any moment, a decision-maker (investor) can either *accept the project* and proceed with the investment or *delay the decision* until he obtains new information;
- investment are considered to be instantaneous and irreversible (they cannot be withdrawn from the project and used for other purposes).

The investor's problem is to evaluate the project and to select an appropriate time for the investment (real option valuing).

There are two different approaches to solving the investor's problem.

1. The value of project is the maximum of Net Present Value (NPV) from the project over all stopping times (regarding to σ-algebras $\mathcal{F}_t$):

$$\max_{\tau} \mathbf{E}(V_\tau - I)e^{-\rho\tau} = \mathbf{E}(V_{\tau^*} - I)e^{-\rho\tau^*}.$$

   An optimal stopping time τ^* is viewed as optimal investment time.
   It is not clear where an arbitrary discount rate ρ should come from.

2. A project is spanned with some traded asset S, which price S_t is completely correlated with present value of the project V_t.
   The value of project is linked with the value of derivative based on this asset S.
   The opportunity to invest is considered as an American style option (to buy the asset on predetermined price I).
   At that a value of option is accepted as a value of investment project, and an exercise time is viewed as the investment time.

Two ways in option valuation

1) Contingent Claims Analysis (CCA).

We have BS-market with risk-free interest rate r and risky asset S, which dynamics $S_t = S_t(\mu)$ is described by geometric Brownian motion (with drift μ and volatility σ), and flow of dividends at rate δ.

On the above BS market we consider a riskless replicated portfolio, and the value of real option is defined as the value of this portfolio under no-arbitrage conditions. In this way the value of option is derived from a solution $(F(s), s^*)$ to the following free-boundary problem :

$$\begin{aligned} &0.5\sigma^2 s^2 F^{''}(s) + (r-\delta)sF^{'}(s) - rF(s) = 0, \quad 0 < s < s^*; \\ &F(s^*) = g(s^*); \\ &F'(s^*) = g'(s^*), \end{aligned} \tag{1}$$

where $g(s) = s - I$.

2) Optimal stopping for American option

We consider on the BS market, defined above, an American option with payoff $f_t = g(S_t) = (S_t - I)^+$. The value of real option is the value of this American option, i.e. $\sup_\tau \mathbf{E}_Q e^{-r\tau} f_\tau$ (over all stopping times τ), and $\mathbf{E}_Q$ is taken at risk-neutral measure Q, such that $\{S_t e^{-(r-\delta)t},\ t \geq 0\}$ is Q-martingale. After the change of measure the value of option can be written as

$$\sup_\tau \mathbf{E} e^{-r\tau} g(S_\tau(r-\delta)), \tag{2}$$

where expectation is taken relative to initial measure $\mathbf{P}$, and risky asset S evolves as geometric Brownian motion with drift $r - \delta$ and volatility σ. Formula (2) for the value of American option holds in more general setting with any payoff function $g(S)$.

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In order to specify the rate of return μ and dividend rate δ of a risky asset S we can embed the BS market into the CAPM model: $\mu = r + \phi\sigma R_{Sm}$, where ϕ is "market price of risk", R_{Sm} is correlation of S with market portfolio; and $\delta = \mu - \alpha$, where α is the expected rate of return of project's present value V_t.

It is commonly accepted that CCA approach

$$0.5\sigma^2 s^2 F''(s) + (r-\delta)sF'(s) - rF(s) = 0, \quad 0 < s < s^*;$$
$$F(s^*) = g(s^*); \quad F'(s^*) = g'(s^*), \tag{3}$$

gives the same solution as the corresponding optimal stopping problem

$$\sup_{\tau} \mathbf{E} e^{-r\tau} g(S_\tau(r-\delta)). \tag{4}$$

This is a case for a classical American call option with the payoff $g(s) = (s - I)^+$, but for the general payoff function a relation between solutions to problems (3) and (4) remains open.

More general question:

What is the connection between optimal stopping problem for diffusion processes and appropriate free-boundary problem?

We study this question in the framework of the variational approach to optimal stopping problems.

The further outline of the talk

- Optimal stopping problem. Free-boundary problem
- Variational approach. One-parametric class of "continuation sets"
- A variational view to a smooth pasting principle
  - A solution to free-boundary problem can not give an optimal stopping
- How a variational approach works.
  - "Russian Option"
  - Optimal investment timing problem under tax exemptions
  - Two-dimensional geometric Brownian motion and non-linear payoff function

Optimal stopping problem

Let $X_t, \ t \geq 0$ be a diffusion process with values in $\mathbb{R}^n$ defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbf{P})$ by the following stochastic equation:

$$dX_t = a(X_t)dt + b(X_t)dW_t, \quad X_0 = x,$$

where $a: \ \mathbb{R}^n \to \mathbb{R}^n$ is vector of drift coefficients,
$b: \ \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is matrix of diffusion,
$W_t = (w_t^1, ..., w_t^n)$ – standard multi-dimensional Wiener process.

Infinitesimal operator of the process X_t :

$$\mathbb{L}_X = \sum_i a_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} \Big[b(x) b^{\mathsf{T}}(x) \Big]_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

(*semi-elliptic* partial differential operator on $C^2(\mathbb{R}^n)$)

Let us consider *an optimal stopping problem* (OSP) for this process:

$$U(x) = \sup_{\tau} \mathbf{E}^x g(X_\tau) e^{-\rho\tau}, \qquad (5)$$

where $g: \mathbb{R}^n \to \mathbb{R}^1$ is *payoff function*, $\rho \geq 0$ is *discount rate*, and $\mathbf{E}^x$ means the expectation for the process X_t starting from the initial state x. The maximum in (5) takes over some class of stopping times (s.t.) τ, usually over the class $\mathcal{M}$ of all stopping times with respect to the natural filtration $(\mathcal{F}_t^X = \sigma\{X_s,\ 0 \leq s \leq t\},\ t \geq 0)$.

Traditional solving of problem (5) is to find stopping time $\tau^*(x)$, at which sup in (5) is attained, as well as the value function $U(x)$, for all initial states x. In other words, (5) is considered as the family of problems depending on the parameter x (*"mass" setting*).

Free-boundary problem

Optimal stopping time for the problem (5) can be represented as the first exit time of the process X_t out of the continuation set $C = \{U(x) > g(x)\}$. Usually it is proposed to find unknown function $U(x)$ and continuation set C as a solution to *free-boundary problem* (Stefan problem):

$$\mathbb{L}_X U(x) = \rho U(x), \quad x \in C; \tag{6}$$

$$U(x) = g(x), \quad x \in \partial C; \tag{7}$$

$$\mathrm{grad}\, U(x) = \mathrm{grad}\, g(x), \quad x \in \partial C \tag{8}$$

where $\mathbb{L}_X$ is the infinitesimal generator of X_t,
∂C is the boundary of the set C.

The condition (7) is called *continuous pasting*,
and (8) – *smooth pasting*.

A proof of necessity of the condition (8) for one-dimensional diffusion processes one can find, e.g., in *A.Shiryaev & G.Peskir (2006)*.

Variational approach

We develop another approach to solving an optimal stopping problem which we shall refer as the variational. We a priori define a class of "*continuation regions*", and we find **the optimal region** over this given class.

Unlike the mass setting of an optimal stopping problem, we study the individual OSP for the given (fixed) initial state of the process $X_0 = x$.

Let $\mathcal{G} = \{G\}$ be a given class of regions in $\mathbb{R}^n$,
$\tau_G = \tau_G(x) = \inf\{t \geq 0 : \; X_t \notin G\}$ be a first exit time of process X_t out of the region G (obviously, $\tau_G = 0$ whenever $x \notin G$), and
$\mathcal{M}(\mathcal{G}) = \{\tau_G, \; G \in \mathcal{G}\}$ be a set of first exit times for all regions from the class $\mathcal{G}$.

We will suppose that $\tau_G < \infty$ (a.s.) for any $G \in \mathcal{G}$, i.e. $\{\tau_G\}$ are stopping times.

Under fixed initial value x for any continuation region $G \in \mathcal{G}$ define a function of sets:

$$V_G(x) = \mathbf{E}^x g(X_{\tau_G}) e^{-\rho \tau_G}. \tag{9}$$

If $x \in G$ then $V_G(x)$ can be derived from solutions to boundary Dirichlet problems:

$$\begin{aligned} \mathbb{L}_X u(x) = \rho u(x), &\quad x \in G; \\ u(x) = g(x), &\quad x \in \partial G. \end{aligned}$$

To calculate functions (9) one can also use martingale methods.

Thus, a solving an optimal stopping problem over a class $\mathcal{M}(\mathcal{G})$ can be reduced to a solving the following variational problem:

$$V_G(x) \to \max_{G \in \mathcal{G}}. \tag{10}$$

If G^* is an optimal region in (10), the optimal stopping time over the class $\mathcal{M}(\mathcal{G})$ is the first exit time from this region: τ_{G^*}.

One-parametric class of "continuation regions"

Under some additional assumptions a general variational problem can be simplified and be made more convenient for study.

Let D be a set of initial states of the process X_t. Let $\mathcal{G} = \{G_p,\ p \in P \subset \mathbb{R}^1\}$ be one-parametric class of regions in $\mathbb{R}^n$, $\tau_p = \inf\{t \geq 0 : X_t \notin G_p\}$, $V_p(x) = V_{G_p}(x)$.

We will call function $F(p,x)$, defined on $P \times D$, *a terminal-initial* function if $F(p,x) = V_p(x)$ for $p \in P, x \in G_p$. Then

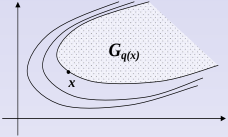
$$V_p(x) = \begin{cases} F(p,x), & x \in G_p \\ g(x), & x \notin G_p \end{cases}.$$

It is assumed that "continuous pasting" holds at the boundary of set G_p:

$$F(p,x) = g(x),\ x \in \partial G_p. \tag{11}$$

Assume that a family of regions $\{G_p\}$ satisfies the conditions:

1. Monotonicity. $G_{p_1} \subset G_{p_2}$ whenever $p_1 < p_2$.
2. "Thickness". Every point $x \in D$ belongs to the boundary of the unique set $G_{q(x)}$. A parameter of those set will be refered as $q(x)$, so $x \in \partial G_{q(x)}$.



Under the above assumptions a maximization of $V_p(x)$ in p can be reduced to a maximization of "simpler" terminal-initial function $F(p, x)$.

Theorem 1

Let for $x \in D$ a terminal-initial function $F(p, x)$ have a unique maximum (in $p \in P$) at the point $p^*(x)$, and $F(p, x)$ decreases in p whenever $p > p^*(x)$. Then $\tau_{p^*(x)}$ is an optimal stopping time in OSP over the class $\mathcal{M}(\mathcal{G})$.

A variational view to a smooth pasting principle

Let D is an open set (in $\mathbb{R}^n$), functions $F(p,x)$, $g(x)$, $q(x)$ (a parameter of the region whose boundary passes through the point x), are differentiable.

Let $\bar{p}(x)$ be a stationary point of $F(p,x)$ in p, i.e. $F'_p(\bar{p}(x),x) = 0$ $(x \in D)$, and $\bar{p}(x) = \bar{p}$ do not depend on x. The continuous pasting implies for the function $\overline{F}(x) = F(\bar{p},x)$ the relation:

$$\mathrm{grad}\, V_{\bar{p}}(x) = \mathrm{grad}\, \overline{F}(x) = \mathrm{grad}\, g(x), \quad x \in \partial G_{\bar{p}}. \tag{12}$$

The equality (12) is a traditional smooth pasting condition, and, therefore, $(\bar{F}(x), G_{\bar{p}})$ is a solution to free-boundary problem (Feynmann-Kac formula).

Smooth pasting is a first-order necessary condition to a stationary point *(from variational point of view)*, i.e. the weakest optimality condition

If $\mathrm{grad}\, q(x) \neq 0$ for all $x \in D$, then a smooth pasting condition (12) is equivalent to stationarity of a terminal-initial function $F(p,x)$ (in p) at the point $\bar{p}$.

Second-order conditions for optimal stopping times

Let $X_t,\ t \geq 0$ be a diffusion process with values in $\mathbb{R}^1$ with infinitesimal operator $\mathbb{L}_X$, and $g \in C^2(\mathbb{R}^1)$.

The class of stopping times $\tau_p = \min\{t \geq 0 : X_t \geq p\}$, $\mathcal{M} = \{\tau_p, p \in \mathbb{R}^1\}$

Let $(U(x), p^*)$ be the solution to free-boundary problem:

$$\begin{aligned} &\mathbb{L}_X U(x) = \rho U(x), \quad x < p^*, \\ &U(p^*) = g(p^*), \\ &U'(p^*) = g'(p^*). \end{aligned}$$

(1) If $U''(p^*) < g''(p^*)$, then τ_{p^*} is not the optimal stopping time (over the class $\mathcal{M}$).

(2) If $U''(p^*) > g''(p^*)$, then τ_{p^*} is *the local* optimal stopping time, i.e. it gives a local maximum (in p) to the variational functional $V_p(x)$.

(3) If $U''(p^*) > g''(p^*)$, and free-boundary problem has *a unique solution*, then τ_{p^*} is the optimal stopping time (over the class $\mathcal{M}$).

A solution to free-boundary problem can not give a solution to optimal stopping problem

Example

One-dimensional geometric Brownian motion
$dX_t = X_t(0.5dt + dw_t)$, $X_0 = x$, (where w_t be standard Wiener process),
payoff function $g(x) = g_\delta(x) = (x-1)^3 + x^\delta$ for $x \geq 0$ $(\delta > 0)$,
discount rate $\rho = \delta^2/2$.
The function $g(x)$ is smooth and increasing (in x) for all $\delta > 0$.

A free-boundary problem for finding unknown function $U(x)$ and boundary p^* is the following one:

$$\begin{cases} \frac{1}{2}x^2U''(x) + \frac{1}{2}xU'(x) = \rho U(x), & 0 < x < p^* \\ U(p^*) = g(p^*), \\ U'(p^*) = g'(p^*). \end{cases}$$

Let $V_p(x)=\mathbf{E}^x g(X_{\tau_p})e^{-\rho\tau_p}$, where $\tau_p = \min\{t\geq 0 : X_t \geq p\}$.

For $\delta < 3$ the free-boundary problem has the unique solution $U(x) = V_1(x) = x^\delta$, $p^* = 1$, but the optimal stopping problem has no solution, since $V_p(x) \to \infty$ when $p \to \infty$ (for all $x > 0$).

For $\delta = 3$ the free-boundary problem also has the unique solution
$U(x) = V_1(x) = x^3$, $p^* = 1$,
but the optimal stopping problem has no solution,
since $V_p(x) \uparrow V(x) = 2x^3$ when $p \to \infty$ (for all $x > 0$).
For this case $V(x) = \sup_\tau \mathbf{E}^x g(X_\tau)e^{-\rho\tau} \neq U(x)$.

For $\delta > 3$ the free-boundary problem has two solutions:
(a) $U(x)=V_1(x)=x^\delta$, $p^*=1$, and
(b) $U(x)=V_{p_\delta}(x)=h(p_\delta)x^\delta$, $p^*=p_\delta=\delta/(\delta-3)$,
but the case (a) does not give a solution to the optimal stopping problem (which there exists, in contrast to the previous case).

Pricing in **"Russian Option"** can be reduced to OSP (with payoff $g(x)=x$) for the diffusion process $(\psi_t, t \geq 0)$ with reflection:
$d\psi_t = -\psi_t(rdt + \sigma dw_t) + d\varphi_t$, where non-decreasing process $(\varphi_t, t \geq 0)$ grows whenever $(\psi_t, t \geq 0)$ attains boundary $\{1\}$ *(L.Shepp & A.Shiryaev)*.

Consider the class of stopping times $\tau_p = \min\{t \geq 0 : \psi_t \geq p\}, \; p > 1$.

Following the explicit formula for $V_p(x) = \mathbf{E}^x \psi_{\tau_p} e^{-\rho\tau_p}$, we can view

$$F(p,x) = p \cdot \frac{\beta_2 x^{\beta_1} - \beta_1 x^{\beta_2}}{\beta_2 p^{\beta_1} - \beta_1 p^{\beta_2}}, \qquad p \geq 1, \quad x \geq 1,$$

where $\beta_1, \; \beta_2$ are roots of the equation $\sigma^2\beta^2 - (\sigma^2 + 2r)\beta - 2\rho = 0$ $(\beta_1 < 0, \; \beta_2 > 1)$, as the terminal-initial function.

$F(p,x)$ attains the unique maximum (in $p \geq 1$) for all $x > 1$ at the point $p^* = \left[\frac{\beta_2(1-\beta_1)}{\beta_1(1-\beta_2)}\right]^{1/(\beta_2-\beta_1)}$, and decreases for $p > p^*$. Thus, Theorem 1 implies that τ_{p^*} is optimal stopping time over the class $\{\tau_p, \, p > 1\}$.

Optimal investment timing problem under tax exemptions

Suppose that investment into creating a firm is made at time $\tau \geq 0$,
I_τ be cost of investment required to create firm at time τ,
$\pi^\tau_{\tau+t}$ be the flow of profits from the firm,
$D^\tau_{t+\tau}$ — the flow of depreciation charges (diminishing the tax base),
γ is the corporate profit tax rate.

A creation of a new firm in real sector of economy is usually accompanied by tax holidays (exemption from profit tax) during the payback period ν_τ:

$$\nu_\tau = \inf\{\nu \geq 0 : \ \mathbf{E}\left(\int_0^\nu \pi^\tau_{\tau+t} e^{-\rho t} dt \,\middle|\, \mathcal{F}_\tau \right) \geq I_\tau\}$$

(if infimum is not attained, then we put $\nu_\tau = \infty$).

The present value of the firm (at the investment time τ) is:

$$V_\tau = \mathbf{E}\left(\int_0^{\nu_\tau} \pi^\tau_{\tau+t} e^{-\rho t} dt + \chi_\tau \int_{\nu_\tau}^\infty [(1-\gamma)\pi^\tau_{\tau+t} + \gamma D^\tau_{t+\tau}] e^{-\rho t}\, dt \,\middle|\, \mathcal{F}_\tau \right),$$

Investment timing problem

The investor's decision problem is to find such a stopping time τ (investment rule), that maximizes the expected NPV from the future firm

$$\mathbf{E}\left(V_\tau - I_\tau\right) e^{-\rho\tau} \to \max_\tau,$$

where the maximum is considered over all stopping times $\tau \in \mathcal{M}$.

Mathematical assumptions

The process of profits $(\pi^{\tau}_{\tau+t},\ t,\tau\geq 0)$ is represented as

$$\pi^{\tau}_{\tau+t} = \pi_{\tau+t}\xi^{\tau}_{\tau+t},\ \ t,\tau \geq 0,$$

where $(\pi_t,\ t \geq 0)$ is geometric Brownian motion:

$$d\pi_t = \pi_t(\alpha_1 dt + \sigma_{11} dw^1_t) \qquad (\pi_0 > 0,\ \sigma_{11} > 0), \quad t \geq 0,$$

and $(\xi^{\tau}_{\tau+t},\ t\geq 0)$ is a family of non-negative diffusion processes $(t,\tau \geq 0)$:

$$\xi^{\tau}_{\tau+t} = 1 + \int_{\tau}^{\tau+t} a(s-\tau,\xi^{\tau}_s)\, ds + \int_{\tau}^{\tau+t} b_1(s-\tau,\xi^{\tau}_s)\, dw^1_s + \int_{\tau}^{\tau+t} b_2(s-\tau,\xi^{\tau}_s)\, dw^2_s,$$

where w^2_t is standard Wiener process independent on w^1_t.

The process π_t can be related to the market prices of goods and resources, whereas fluctuations $\xi^{\tau}_{\tau+t}$ can be generated by the firm created at time τ (firm's uncertainty).

We will suppose that $\mathbf{E}\pi^{\tau}_{\tau+t} < \infty$ for all $t,\tau \geq 0$.

The cost of the required investment I_t is also the geometric Brownian motion:

$$dI_t = I_t(\alpha_2 dt + \sigma_{21} dw_t^1 + \sigma_{22} dw_t^3), \quad (I_0 > 0) \quad t \geq 0,$$

where standard Wiener process w_t^3 is independent on w_t^1 , w_t^2, and $\sigma_{21} \geq 0,\ \sigma_{22} > 0$.

The flow of depreciation charges at time $t+\tau$ will be represented as

$$D^{\tau}_{\tau+t} = I_\tau a_t, \quad t \geq 0,$$

where $(a_t,\ t{\geq}0)$ is the "depreciation density", characterizing a depreciation policy, i.e. non-negative function $a : \mathbb{R}^1_+ \to \mathbb{R}^1_+$, such that $\int_0^\infty a_t\, dt = 1$.

Reduction to optimal stopping problem

The investment timing problem

$$\mathbf{E}\left(V_\tau - I_\tau\right) e^{-\rho\tau} \to \max_{\tau\in\mathcal{M}}, \tag{13}$$

is reduced to optimal stopping problem for geometric Brown.motion (π_t, I_t):

$$\mathbf{E} g(\pi_\tau, I_\tau) e^{-\rho\tau} \to \max_{\tau\in\mathcal{M}}, \tag{14}$$

with homogeneous payoff function $g(\pi, I) = (1-\gamma)(\pi B - I) + \gamma I A(\nu(\pi/I))$,
where $A(\nu) = \int_\nu^\infty a_t e^{-\rho t} dt$, $\quad \nu(p) = \min\{\nu > 0 : \int_0^\nu B_t e^{-\rho t} dt \geq p^{-1}\}$,

$$B_t = \mathbf{E}(\pi_t \xi_t^0)/\pi_0, \qquad B = \int_0^\infty B_t e^{-\rho t}\, dt < \infty.$$

If τ^* is an optimal stopping time for (14) and $\nu(\pi_{\tau^*}/I_{\tau^*}) < \infty$ (a.s.), then τ^* is the optimal investment time for the problem (13).

Two-dimensional geometric Brownian motion and non-linear payoff function

We can apply the variational approach to optimal stopping problem for two-dimensional geometric Brownian motion $X_t = (X_t^1, X_t^2),\ t \geq 0$

$$\begin{aligned} dX_t^1 &= X_t^1(\alpha_1 dt + \sigma_{11} dw_t^1 + \sigma_{12} dw_t^2), \qquad & X_0^1 &= x_1, \\ dX_t^2 &= X_t^2(\alpha_2 dt + \sigma_{21} dw_t^1 + \sigma_{22} dw_t^2), \qquad & X_0^2 &= x_2, \end{aligned} \tag{15}$$

where (w_t^1, w_t^2) is standard two-dimensional Wiener process.

Let payoff function $g(x_1, x_2)$ be continuous and positive homogeneous of order $m \geq 0$, i.e. $g(\lambda x) = \lambda^m g(x)$ for all $\lambda > 0$, $x_1, x_2 \geq 0$.

The region of the initial states of X_t is $D = \{(x_1, x_2) : x_1, x_2 > 0\}$, and $G_p = \{(x_1, x_2) \in D : \ x_1 < p x_2\},\ p > 0$ are continuation sets.

$\tau_p(x) = \min\{t \geq 0 : \ X_t^1 \geq p X_t^2\}$ denotes the first exit time of the process (15) from the region G_p.

If the following (standard) conditions hold

$$\alpha_1 - \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2) \geq \alpha_2 - \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2), \tag{16}$$

$$\rho > \max(\bar{\alpha}_1, \bar{\alpha}_2)m, \text{ where } \bar{\alpha}_i = \alpha_i + \frac{1}{2}(m-1)(\sigma_{i1}^2 + \sigma_{i2}^2), \ i = 1, 2. \tag{17}$$

then the function $V_p(x) = \mathbf{E}^x e^{-\rho\tau_p(x)} g(X_{\tau_p(x)})$ is the solution to Dirichlet problem and has the following type:

$$V_p(x_1, x_2) = h(p) x_1^{\beta} x_2^{m-\beta} \qquad (\text{if } 0 < x_1 < p x_2),$$

where $h(p) = g(p, 1) p^{-\beta}$ and β is a positive root of the quadratic equation

$$\frac{1}{2}\tilde{\sigma}^2 \beta(\beta - 1) + \left(\bar{\alpha}_1 - \bar{\alpha}_2 - \frac{m-1}{2}\tilde{\sigma}^2\right)\beta - (\rho - \bar{\alpha}_2 m) = 0,$$

where $\tilde{\sigma}^2 = (\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2$.

As a terminal-initial function $F(p,x)$ for the considered OSP we can take

$$F(p,x) = h(p)x_1^{\beta}x_2^{m-\beta}, \quad h(p) = g(p,1)p^{-\beta}. \tag{18}$$

Maximum of the function $F(p,x)$ in p is attained at the same point p^* as maximum of $h(p)$, i.e. this point does not depend on x.

The class $\{G_p,\ p > 0\}$ satisfies the requirements of monotonicity and thickness for continuation regions, and τ_p are stopping times. Thus, applying Theorem 1 to the optimal stopping problem we obtain

Theorem 2

Let standard conditions (16), (17) hold, $\tilde{\sigma} > 0$, p^* be the unique maximum point of the function $h(p)$, defined in (18), and $h(p)$ decreases for $p > p^*$. Then $\tau_{p^*} = \min\{t \geq 0 :\ X_t^1 \geq p^* X_t^2\}$ is optimal stopping time over the class $\{\tau_p,\ p>0\}$.

(For the linear payoff function $g(x_1,x_2)$ the conditions on $h(p)$ hold surely)

The class of continuation regions $\{G_p,\ p>0\}$ for the considered problem is chosen "well" and τ_{p^*} will be also optimal (under additional assumptions) over the class of *all stopping times.*

Theorem 3

Let all conditions of Theorem 2 hold, $g \in C^2(\mathbb{R}^2_+)$, $p^*>0$ be the unique maximum point of the function $h(p)$ and $g'_{x_1}(p,1)p^{-\beta+1}$ decreases for $p>p^*$. Then $\tau^*=\min\{t\geq 0:\ X^1_t \geq p^* X^2_t\}$ is optimal stopping time over the class of all stopping times.

Corollary *(McDonald & Siegel, Hu & Øksendal)*

Let $g(x_1,x_2)=c_2x_2-c_1x_1$ $(c_1,c_2>0)$, $\tilde{\sigma}>0$, condition (16) hold, and $\rho>\max(\alpha_1,\alpha_2)$. Then the optimal stopping time (over all stopping times) is $\tau^*=\min\{t\geq 0:\ X^1_t \geq p^* X^2_t\}$, where $p^*=c_1c_2^{-1}\beta(\beta-1)^{-1}$, and β is a positive root of the equation $\frac{1}{2}\tilde{\sigma}^2\beta(\beta-1)+(\alpha_2-\alpha_1)\beta-(\rho-\alpha_1)=0$.

In order to prove the optimality of stopping time τ^* over the class of all stopping times we use the following "verification theorem", based on variational inequalities method.

Verification Theorem *(Øksendal)*

Suppose, there exists a function $\Phi : \mathbb{R}^n_+ \to \mathbb{R}^1$, satisfying the following conditions:

1) $\Phi \in C^1(\mathbb{R}^n_+)$, $\Phi \in C^2(\mathbb{R}^n_+ \setminus \partial\Gamma)$; where $\Gamma = \{x \in \mathbb{R}^n_+ : \ \Phi(x) > g(x)\}$,
2) $\partial\Gamma$ is locally the graph of Lipschitz function and $\mathbf{E}^x \displaystyle\int_0^\infty \chi_{\partial\Gamma}(X_t)\, dt = 0$ for all $x \in \mathbb{R}^n_+$;
3) $\Phi(x) \geq g(x)$ for all $x \in \mathbb{R}^n_+$;
4) $\mathbb{L}\Phi = \rho\Phi$ for $x \in \Gamma$;
5) $\mathbb{L}\Phi \leq \rho\Phi$ for $x \in \mathbb{R}^n_+ \setminus \bar{\Gamma}$ ($\bar{\Gamma}$ is a closure of the set Γ);
6) $\bar{\tau} = \inf\{t \geq 0 : \ X_t \notin \Gamma\} < \infty$ a.s. for all $x \in \mathbb{R}^n_+$;
7) the family $\{g(X_\tau)e^{-\rho\tau}, \ \tau \leq \bar{\tau}\}$ is uniformly integrable for all $x \in \Gamma$.

Then $\bar{\tau}$ is an optimal stopping time (over all stopping times), and $\Phi(x)$ is the value function.

Return to investment timing problem

Let β be a positive root of the quadratic equation

$$\frac{1}{2}\tilde{\sigma}^2\beta(\beta-1)+(\alpha_1-\alpha_2)\beta-(\rho-\alpha_2)=0, \qquad \tilde{\sigma}^2=(\sigma_{11}-\sigma_{21})^2+\sigma_{22}^2.$$

Then Theorem 3 implies

Theorem 4

Let $a_t, B_t \in C^1(\mathbb{R}_+)$ and all conditions of Theorem 3 hold. Then the optimal investment time for the investment timing problem (13) is $\tau^* = \min\{t \geq 0 : \pi_t \geq p^* I_t\}$, where p^* is a root of the equation

$$\beta(1-\gamma) + \frac{\gamma a_{\nu(p)}}{pB_{\nu(p)}} = (1-\gamma)(\beta-1)pB + \beta\gamma A(\nu(p)).$$

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