Economic Default vs. Default

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Modeling default: structural models

- In 1974, Merton introduced the first structural model using the Black and Scholes theory on option pricing.
- In 1976, Black and Cox introduced the first credit risk model using the first hitting time below a barrier; This barrier can be set exogenously, or endogenously to satisfy the shareholders.
- Many people tried to improve structural models, adding jumps at the diffusion for instance.
- However, most of the time short term credit spreads are underestimated.

Modeling default: intensity-based models

- These models do not consider the relation between default and firm value in an explicit manner.
- Default comes as a 'surprise'.
- There is an intensity of the arrival of default λ_t.
- λ_t is an exogenous random process.
- Default can be not predicted: totally inaccessible.

Modeling default: latest models

- Noisy partial information: (Duffie and Lando, 2001) The information (filtration) is not the same for the investors and for the firm managers. Hence the spread in strictly positive when $T \to 0$.
- Delayed information models (Schonbucher 2003, Guo Jarrow and Zeng 2007, Collin-Dufresne et al. 2002)
- Information-based models are a mixture of the structural models and the reduced-form models.

Reality check

Recently, an empirical study conducted by Guo, Jarrow and Lin on 2500 defaulted bonds showed that:

- The market anticipates the default before it actually occurs.
- They proposed a new definition: The ***Economic default*** as the first time the market prices the debt as if it defaulted.
- Their *statistical* definition:

$$\tau_e = \inf_{t \leq \tau} \{ t : B_t \leq B_\tau^d e^{-\int_t^\tau r_s ds} \},$$

  given τ, where τ is the real default.
- Using economic default, the average pricing error between economic default and recorded default is less than one basis point.

Economic default

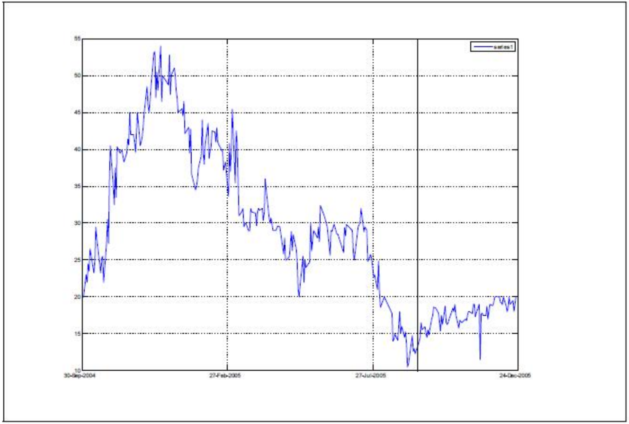


Figure 1: Delta Airlines Inc., coupon 8.3%, maturity 12/15/2029. Time series graph of debt prices as a percentage of face value ($100). The solid vertical line represents the default date.

Economic default

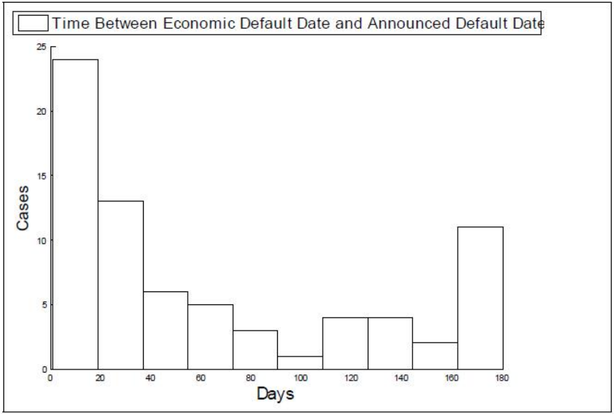


Figure 5: Histogram of the Time Between the Economic and Recorded Default Dates. The number of nonzero differences is 73.

Debt and equity structure of the firm

- In our model, the firm needs to pay back its debt at discrete times $T_1, T_2, ..., T_n$ where for all $k \geq 1$, $T_k = kT$ for some $T > 0$ (T being the time between two consecutive pay back time for instance: 1 week, 3 months....).
- For each of these times, the amount of debt that has to be paid pack is D_k.
- To simplify we assume that $D_{k+1} - D_k$ is constant over time. This is the case when the firm is rolling over its debt with a constant spread.
- The value of the firm $(S_t, t \geq 0)$ follows a geometric Levy process.

Default and economic default

As for structural models, we set for all time T_k a threshold B_k function of the whole sequence $(D_k, k \geq 0)$. For instance:

- $B_k = D_k$ if we only focus on the short term debt.
- B_k can be the whole amount of debt. This is the threshold for the classic Merton model.
- Similar to the KMV model, B_k can be equal to D_k (short term debt) + a fraction of long term debt.

Default and economic default

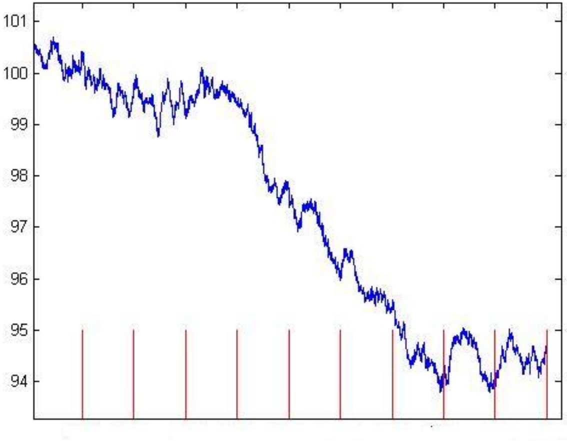
- In our model, the default occurs when the asset value of the firm is less that the threshold B_k at a time T_k. The default can only happen for some time T_k:

$$\tau = \inf\{T_k, V_{T_k} < B_k\}.$$

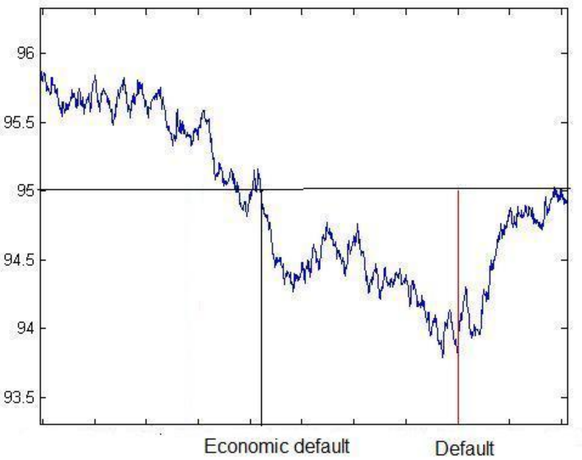
- *Mathematically*, the economic default is defined as the last time the asset value of the firm is above this threshold:

$$\tau_e = \sup\{t > T_{\tau-1}, V_t \geq B_\tau\}.$$

Our model



Our model



The law of the real default τ?

For all $n \geq 0$, $x \geq 0$ we want to compute

$$u_n(x) = \mathbb{P}[\tau = T_n | S_0 = x, T, B_1, B_2, \ldots].$$

- If for all $k \geq 0$, $B_k = B$ is constant (we have the same threshold for all times T_k), τ is the first hitting time under $\log(B)$ of the random walk $(M_i = \log(S_{iT}), i \geq 0)$.
- If $(B_{k+1} - B_k, k \geq 1)$ is constant, τ is the first hitting time under $\log(B)$ of the random walk $(M_i = \log(S_{iT}) - (B_{k+1} - B(k))/T, i \geq 0)$.

Law of the real default τ

Using the fluctuation theory of random walks we prove that:

Theorem

$$\mathbb{P}[\tau = k, S_\tau \in du | S_0 = x, T, B_1, B_2, ...] =$$

$$\int_0^\infty \int_0^y \sum_{i=0}^{k-1} \mathbb{U}^-(\log(x/B) - dy, k-i)\mathbb{U}^+(dv - y, i)F(-v - du),$$

- where $\mathbb{U}^+$ (resp $\mathbb{U}^-$) is the green function of the increasing (resp decreasing) ladder process (cf: fluctuation theory) of the random walk $(M_i, i \geq 0)$.
- F is the cdf of M_1.

Property of the credit spread

- Using the same method we can compute

$$u_n(t,x) = \mathbb{P}[\tau = T_n | S_t = x, T, B_1, B_2, ...].$$

- The credit spread is a function of both the equity dynamics and the debt structure.
- Depending on the dynamics of the asset value $(S_t, t \geq 0)$, and on the tressholds $(B_k, k \geq 1)$, we can generate increasing, decreasing or bumped credit spreads.
- For a general debt structure $((T_k, D_{T_k}), k \geq 1)$, we can't use the fluctuatuion theory of random walks. We have to do Monte Carlo simulations to compute the sequence $(u_n(x,t), n \geq 1)$.

Law of economic default

Technical difficulties:

- This is not a stopping time.
- This is a last passage time and not a first passage time.
- It is defined conditioned on the knowledge of the real default τ.

Main idea

We first compute the distribution of the time between the economic default and the real default $\tau - \tau_e$ conditioned on the default time $\tau = T_n$ and then get the unconditioned law by summing over all T_n and using the Markov property of Levy processes.

The law of τ_e conditioned on τ

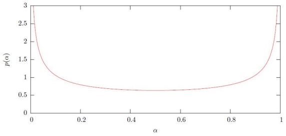
When the dynamics of the asset value $(S_t, t \geq 0)$ is continuous:

$$\mathbb{P}[\tau - \tau_e \in ds | S_{T_{k-1}} = x, \tau = T_k] = \int_0^T \frac{\mathcal{A}_u(s)}{\mathbb{P}[H_{B-x} \leq T]} \phi(u) du.$$

- H_{B-x} is the first hitting time under $B - x$ of a stochastic process with law $(S_t, t \geq 0)$ starting from 0. $\phi(u)$ is its pdf.
- $\mathcal{A}_T$ is the pdf of the last passage at zero time before T of a stochastic process with law $(S_t, t \geq 0)$ starting from 0. This law is the arcsine law if $(S_t, t \geq 0)$ is a Brownian motion.

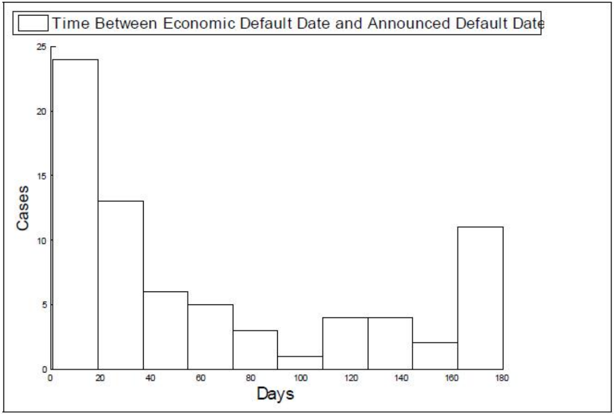
Arcsine law of $\tau - \tau_e$

- The unconditioned law of $\tau_e - \tau$ is obtained by summing over all $n \geq 1$.
- Both conditioned and unconditioned distribution of the time between economic default and real default follow a mixture of arcsine law (U-shaped).



- We conjecture that this is still true for jump-diffusion Levy processes.

Economic default vs Default



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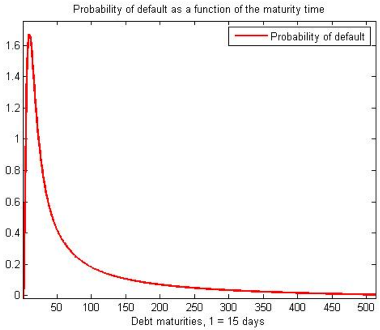
Example 1



Here the dynamics of the firm is a geometric Brownian motion, $\sigma = 0.25$, $r = 0.04$. Leverage = 0.8. T = 15 days. $B_1 = B_2 = 0.4$. $D_1 = D_2 = 0.8$ (as in KMV; all the debt is short term)

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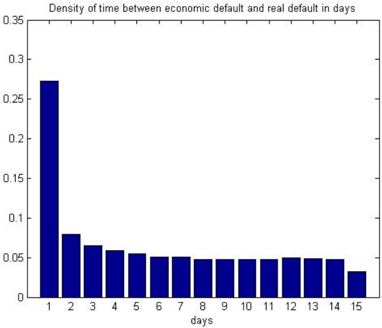

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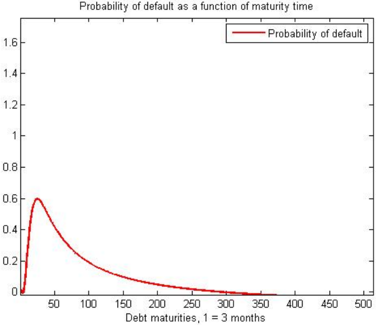
Example 2



The dynamics of the firm is a geometric Brownian motion, $\sigma = 0.25$, $r = 0.04$. Leverage = 0.8. T = 3 months. $D_1 = \ldots = D_8 = 0.1$. $B_1 = \ldots = B_8 = 0.4$ (short term + 1/3 of long term debt).

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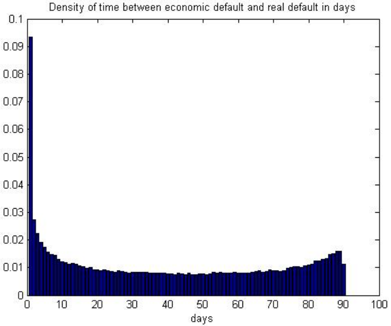

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Conclusion

- New mathematical concept and model for economic default.
- Quantitative model consistent with empirical observation.
- Analytical formula for distribution of τ and τ_e.
- Arcsine law of the distribution of $\tau - \tau_e$.
- Credit spread is a function of the dynamics of the value of the firm *and* of the debt structure of the firm,

Thank you for your attention.