Risk management, Arbitrage and Scenario generation for interest rates

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Toronto, June 2010

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- Generated scenarios should share the most important stylized facts of the respective time series of risk factor (mean, covariance, skewness, kurtosis, heavy tails, stochastic volatility, etc) in order to reflect the market's information.
- The actual generation of scenarios must be quick (up to one hour) and flexible (changes of markets should be directly implemented into the scenario generator).

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- Usually the number of risk factors N is (substantially) larger than the number of observations K.

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- Control the generation of scenarios by stochastic differential equations, which **describe the local dynamics of the respective risk factors** and which are standard models in finance.
- Impose no arbitrage conditions.
- a robust and quick calibration method (no optimization!).

Why No Arbitrage?

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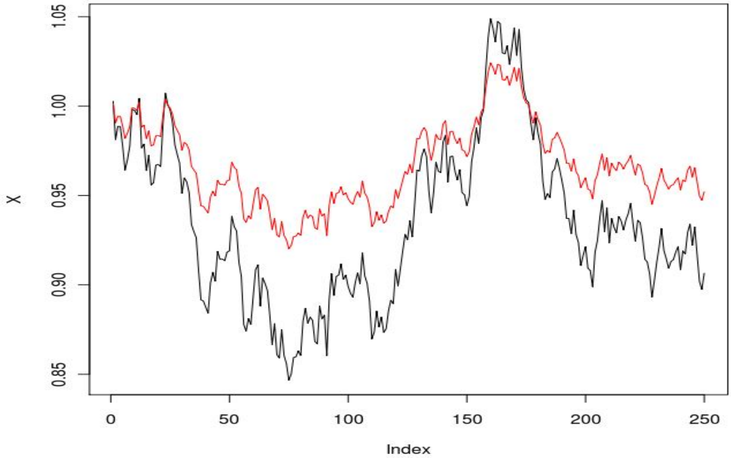
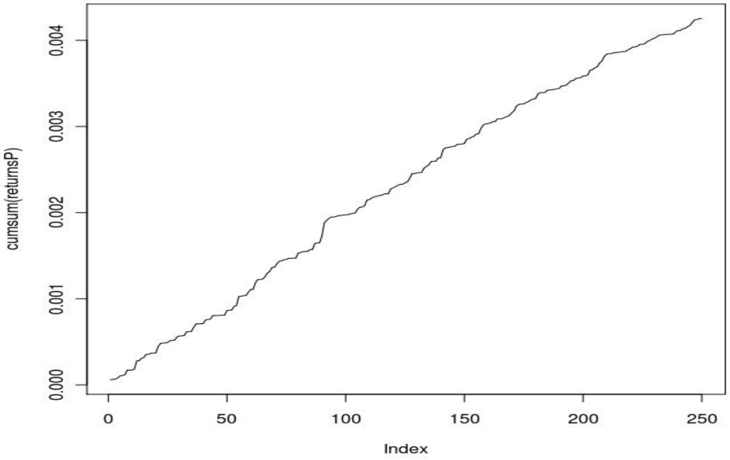
- calculate the returns of the log prices and estimate the covariance matrix.
- suppose that to the best of your knowledge you observe Gaussian returns with perfect correlation $\rho = 1$ and standard deviations 0.2 and 0.1.
- simulate those distributions and reconstruct a trajectory.

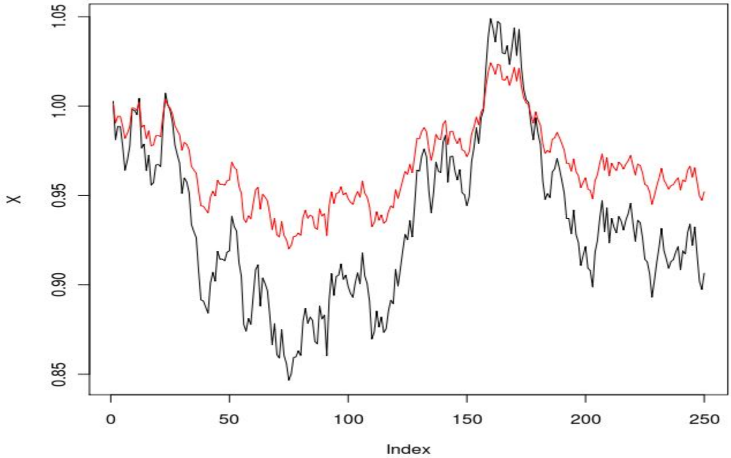
Simulated stock price processes



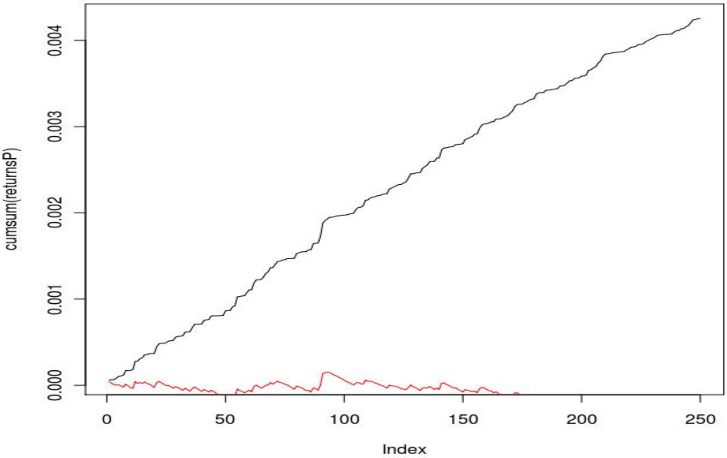
P & L of portfolio σ_Y/σ_X long in X and short in Y



Simulated stock price processes without arbitrage



P & L for the same portfolio with or without arbitrage



Risk management, Arbitrage and scenario generation

In terms of stochastic analysis the implemented example is

$$dX_t = 0.02X_t dt + 0.2X_t dB_t,\ dY_t = 0.005Y_t dt + 0.1Y_t dB_t,$$

where we see a clear violation of the no arbitrage condition.
The example is *not artificial* since risk factor models of high dimension are likely to have singular covariance matrices, hence the appropriate drift conditon matters.

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- Risk factors are yield curves.
- The yield curves evolves according to (jump diffusion) HJM or BH equations.
- A robust calibration method of the HJM equation is needed.
- Numerics of SPDEs (Euler, Ninomiya-Victoir).

Interest Rate mechanics

Prices of T-bonds are denoted by $P(t, T)$. Interest rates are governed by a market of (default free) zero-coupon bonds modelled by stochastic processes $(P(t, T))_{0 \leq t \leq T}$ for $T \geq 0$. We assume the normalization $P(T, T) = 1$.

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describes the compound interest rate p.a. for maturity T.
- f is called the *forward rate curve* of the bond market
$$P(t,T)=\exp\left(-\int_t^T f(t,s)ds\right)$$
for $0\leq t\leq T$.

Interest Rate mechanics

- The *short rate process* is given through $R_t = f(t,t)$ for $t \geq 0$ defining the "bank account process"

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- No arbitrage is guaranteed if on the filtered probability space $(\Omega, \mathcal{F}, Q)$ with filtration $(\mathcal{F}_t)_{t\geq 0}$,

$$E(\exp(-\int_t^T R_s ds)|\mathcal{F}_t) = P(t,T)$$

holds true for $0 \leq t \leq T$ for some equivalent (martingale) measure $P \sim Q$. We write $E = E_P$.

HJM-drift condition

The forward rates $(f(t,T))_{0 \leq t \leq T}$ are best parametrized through

$$r(t,x) := f(t,t+x)$$

for $t, x \geq 0$ (Musiela parametrization). No-Arbitrage is guaranteed in a diffusion setting if the HJM-equation

$$dr_t = (\frac{d}{dx} r_t + \sum_{i=1}^{d} \sigma^i(r_t) \int_0^{\cdot} \sigma^i(r_t))dt + \sum_{i=1}^{d} \sigma^i(r_t) dB_t^i$$

describes the time-evolution of the term structure of interest rates with respect to a martingale measure P.

HJM equation as SPDE

We take a Hilbert space of foward rate curves, where point evaluation is possible and where the shift acts as strongly continuous semigroup, then we can understand the HJM equation as SPDE

$$dr_t = (\frac{d}{dx} r_t + \sum_{i=1}^{d} \sigma^i(r_t) \int_0^{\cdot} \sigma^i(r_t))dt + \sum_{i=1}^{d} \sigma^i(r_t) dB_t^i$$

with state space H and initial values $r_0 \in H$.

A conceptual problem

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- Controlling positivity of the short rate proces is therefore the infinite dimensional problem of leaving the set $\{r(0) \geq 0\}$.
- Even if the one solves the problem of positivity the numerical implementation might lead to negative rates.

The Brody-Hughston (BH) approach

We consider a different state space for the time evolution of interest rates. Observe that a positive short rate should lead to the following two stylized facts

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- The prices $P(t, T)$ are decreasing in T.
- The limit for $T \to \infty$ should be 0.
- This suggests to parametrize

$$P(t, T) = \int_{T-t}^{\infty} \rho(t, u) du,$$

where $u \mapsto \rho(t, u)$ is a probability density on $\mathbb{R}_{\geq 0}$.

The BH equation

We can guarantee no arbitrage in a diffusion setting if the BH equation holds

$$d\rho_t = (\rho_t(0)\rho_t + \frac{d}{dx}\rho_t)dt + \sum_{i=1}^{d}(\sigma_i(\rho_t) - \overline{\sigma_i(\rho_t)})\rho_t dW_t^i$$

describes the time evolution of densities ρ_t with respect to a martingale measure P. Here we apply the geometric notation

$$\overline{\sigma_i(\rho)} = \int_0^\infty \sigma_i(\rho)(u)\rho(u)du$$

such that we can guarantee that the vector fields

$$\rho \mapsto (\sigma_i(\rho) - \overline{\sigma_i(\rho)})\rho$$

lie in the tangent space of "Wasserstein space".

HJM versus BH

- The short rate is always positive for solutions of the BH equation.
- It is much (sic!) easier to incorporate jumps in the BH equation than in the HJM equation.
- The BH equation is "more" non-linear since the tangent spaces are ρ-dependent.

Basic situation

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ we consider a d-dimensional standard Brownian motion B. Let H denote a Hilbert space of risk factors, then we consider

$$dY_t = (\mu_1(Y_t) + \mu_2(Y_t))dt + \sum_{i=1}^{d} \sigma(Y_t) \bullet \lambda_i dB_t^i, \tag{1}$$

$$Y_0 \in H, \tag{2}$$

where

$$\sigma(Y) : H_0 \to H_0$$

is an invertible, linear map on the set of **return directions** depending in a Lipschitz way on Y.

The vector fields μ_1 corresponds to the volatility free situation (deterministic drift). The vector field μ_2 corresponds to the no arbitrage drift condtion due to the presence of stochastics and possibly to some change of measure term. The "(stochastic) volatility factor" σ is chosen appropriately for the respective risk factors.

Robust calibration technique

We assume a time series, i.e. an observation of equation (4), on equidistant grid points of distance Δ, denoted by $Y_1, \ldots, Y_K$ and we ask the simple question if we can estimate the volatility directions $\lambda_1, \ldots, \lambda_d$ out of the observations $Y_1, \ldots, Y_K$ in a simple way – given the geometric factor σ?
We announce now an equation of type (4) "calibrated" to the time series

$$
\begin{aligned}
dX_t^{(K)} =& (\mu_1^{(K)}(X_t^{(K)}) + \mu_2(X_t^{(K)}))dt + \\
&+ \frac{1}{\sqrt{\Delta(K-1)}} \sum_{i=1}^{K-1} \sigma(X_t^{(K)}) \bullet (\sigma(Y_i)^{-1}(Y_{i+1} - Y_i))\, dW_t^i,
\end{aligned} \tag{3}
$$

where σ is a the known, non-vanishing geometric factor on the risk factors describing the local dynamics.

Theorem

Let equation (4) be given in the sense that σ and $Y_0 \in \mathrm{dom}(\mathcal{A})$ are given maps, but $\lambda_1, \ldots, \lambda_d$ are unknown. We collect a time series of observations $Y_1, \ldots, Y_K$ on an equi-distant grid of time distance Δ on an interval of length $T = K\Delta$. Refining the observations through $\Delta = \frac{T}{K}$ leads to the following limit theorem

$$\lim_{K \to \infty} X_t^{(K)} = Y_t$$

in distribution for any $t \geq 0$ if $X_0 = Y_0$.

Theorem

The underlying limit theorem is the following Gaussian one,

$$\begin{aligned}
&\lim_{K\to\infty}\int_0^t \sigma(X_t^{(K)})^{-1} dX_t^{(K)} - \\
&- \int_0^t \sigma(X_s^{(K)})^{-1}(\mu_1^{(K)}(X_s^{(K)}) + \mu_2(X_s^{(K)})ds \\
= &\lim_{K\to\infty} \frac{1}{\sqrt{\Delta(K-1)}} \sum_{i=1}^{K-1} (\sigma(Y_i)^{-1}(Y_{i+1} - Y_i))W_t^i \\
= &\int_0^t \sigma(Y_t)^{-1} dY_t - \int_0^t \sigma(Y_s)^{-1}(\mu_1(Y_s) + \mu_2(Y_s)ds.
\end{aligned}$$

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$$dY_t = (\mu_1(Y_t) + \tilde{\mu_2}(Y_t))dt + \sum_{i=1}^{d} \sigma(Y_t) \bullet \lambda_i \circ dB_t^i, \tag{4}$$

$$Y_0 \in H, \tag{5}$$

where

$$\sigma(Y) : H_0 \to H_0$$

is an invertible, linear map on the set of **return directions** depending in a Lipschitz way on Y. Notice the Stratonovich form.

Geometric integrators

We name the following geometric integrators of the SPDEs 4:

- Evolution in the volatility free direction μ_1 is denoted by FI_1.

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- Evolution in the volatility free direction μ_1 is denoted by FI_1.
- Evolution in directions given by Stratonovich corrected NA-drift terms is denoted by FI_2.
- Evolution along volatilities is denoted by Σ_i.

Euler scheme

In this setting the local step for the Euler scheme reads like

$$Y \mapsto \mathrm{Fl}_1(\Delta) \circ \mathrm{Fl}_2(\Delta)(Y)$$

and

$$Y \mapsto \Sigma_i(\Delta B^i)(Y)$$

for each volatility direciton i. This step is of weak order 2.

Ninomiya-Victoir scheme

The local step is the arithmetic mean of

$$Y \mapsto \mathrm{Fl}_1(\Delta/2) \circ \mathrm{Fl}_2(\Delta/2) \circ \Sigma_1(\Delta B^1) \dots \Sigma_d(\Delta B^d) \circ \mathrm{Fl}_1(\Delta/2) \circ \mathrm{Fl}_2(\Delta/2)$$

and

$$Y \mapsto \mathrm{Fl}_1(\Delta/2) \circ \mathrm{Fl}_2(\Delta/2) \circ \Sigma_d(\Delta B^d) \dots \Sigma_1(\Delta B^1) \circ \mathrm{Fl}_1(\Delta/2) \circ \mathrm{Fl}_2(\Delta/2).$$

This step is of weak order 3.

Then the BH-equation

$$d\rho_t = \left(\rho_t(0)\rho_t + \frac{d}{dx}\rho_t\right)dt + \sum_{i=1}^{d}\left(\sigma_i(\rho_t) - \overline{\sigma_i(\rho_t)}\right)\rho_t dW_t^i$$

$$\rho_0 \in H,$$

has a unique mild solution for all times in H, which leaves the set of densities invariant. Remark the simplicity of the equation with respect to added jumps.

The Stratonovich formulation of the previous equation is

$$
\begin{aligned}
d\rho_t = & \left(\rho_t(0)\rho_t + \frac{d}{dx}\rho_t\right) dt + \\
& - \frac{1}{2}\sum_{i=1}^{d}\left(\sigma_i(\rho_t)^2 - \overline{\sigma_i(\rho_t)^2}\right)\rho_t dt - \\
& - \frac{1}{2}\sum_{i=1}^{d}\left(\eta_i(\rho_t) - \overline{D\eta_i(\rho_t)}\right)\rho_t dt + \\
& + \sum_{i=1}^{d}\left(\sigma_i(\rho_t) - \overline{\sigma_i(\rho_t)}\right)\rho_t dW_t^i, \\
\rho_0 \in & H,
\end{aligned}
$$

where we applied the notation

$$\eta_i(\rho) := D\sigma_i(\rho) \bullet ((\sigma_i(\rho) - \overline{\sigma_i(\rho)})\rho).$$

The flow Fl_1 without volatility

First we solve the SPDE without any noise, which corresponds to a non-homegenous term structure. Assume that all volatility vector fields vanish, then

$$\mathrm{Fl}_1(t)(\rho)(x) = \frac{\rho(t+x)}{\int_t^\infty \rho_0(u)du} \tag{6}$$

as long as $\int_t^\infty \rho_0(u)du > 0$, otherwise the solution vanishes.

Flows in volatility directions

Then we solve the SPDE along noise directions: Let $\xi \in H$, then we can solve

$$d\rho_t = (\xi - \overline{\xi})\rho_t dt \tag{7}$$

explicitly on the space of densities through

$$\rho_t(x) = \frac{\exp(\xi(x)t)\rho_0(x)}{\int_0^\infty \exp(\xi(u)t)\rho_0(u)du} \tag{8}$$

for $x \geq 0$ and $t \in \mathbb{R}$.

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- with arbitrage-free underlying models.
- a robust calibration method to time series data.
- (high-order) numerical schemes for the numerical evaluation.
- easy extensions towards more risk factors, stochastic volatility, jump diffusion, etc.

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