Symmetric periodic orbits in the isosceles three body problem

Daniel C. Offin
Queen's University, Kingston, Canada

We study subsystems of the N-body problem, and give an analytic proof of the existence of hyperbolic periodic orbits which realize certain symbolic sequences of rotations and oscillations in the isosceles three body problem. The Maslov index of the orbits in the reduced space is used to verify hyperbolicity of the minimizing periodic curves in the phase space.

The N-body problem configuration $\mathbf{r} = (r_1, \dots, r_N)$ describes spatial positions of N masses $m_1, \dots, m_N$ where $r_i \in \mathbb{R}^3$. Interaction between the masses is determined by the Newtonian potential function $V(\mathbf{r}) = -\sum_{i<j} \frac{m_i m_j}{\|r_i - r_j\|}$ on the set of non-collision configurations (where $r_i \neq r_j, \quad i \neq j$). The hamiltonian and equations of motion:

$$H(r_1, \dots, r_N, p_{r_1}, \dots, p_{r_N}) = \sum_i^N \frac{1}{2m_i} \|p_{r_i}\|^2 + V(\mathbf{r}).$$

$$\dot{r}_i = \frac{1}{m_i} p_{r_i} \; \dot{p}_{r_i} = \frac{\partial U(\mathbf{r})}{\partial r_i}, \quad i = 1, \dots, N \qquad (1)$$

The configuration manifold modulo translations

$$M = \left\{ \mathbf{r} = (r_1, \dots, r_N) | \sum m_i r_i = 0, r_i \neq r_j, \quad i \neq j \right\}. \tag{2}$$

Angular momentum: $\mathbf{J} = \sum^N r_i \times p_{r_i}$. The angular momentum level set $\mathbf{J}^{-1}(c)$ is invariant under the flow, and the reduced space $\mathbf{J}^{-1}(c)/G_c$, where $G_c = SO(2)$, $c \neq 0$ is the subgroup of $SO(3)$ which fixes $\mathbf{J}^{-1}(c)$, is a symplectic space equipped with the flow of the reduced Hamiltonian vector field.

Relative Equilibria: q_0 is a *central configuration* if q_0 is a critical point of $U(q)$ restricted to the level set $I^{-1}(\frac{1}{2})$, where the *moment of inertia*

$$I(q) = \sum^N m_i \langle r_i, r_i \rangle = \mathbf{r}^2.$$

Solutions to the *N-body equations*

$$q(t) = a(t) q_0, \quad a(t) \in \mathbb{C}, \quad q_0 \in M$$

$$\ddot{a} = -\frac{a}{|a|^3} \frac{U(q_0)}{I(q_0)} = -\frac{a}{|a|^3} \frac{U(q_0)}{\mathbf{r}^2}$$

Variational properties: The action functional for absolutely continuous T-periodic $q(t) \subset M$

$$\mathcal{A}_T(q) = \int_0^T \sum \frac{1}{2m_i} \|p_{r_i}\|^2 + U(\mathbf{r}) dt, \qquad (3)$$

Since the action $\mathcal{A}_T(q_n) < \mathbf{B} < \infty$ where $q_n \in \Lambda$ tends to collision (Sundman 1908), collision curve on the boundary of the functional space Λ may provide the minimizing loop.

Kepler problem: Gordon found that for fixed period T, the minimizing solutions are arranged in families of ellipses, parmeterized by eccentricity e, including the degenerate ellipse when eccentricity $e = 1$ (homothetic collision-ejection path).

For the N-body problem, the action of the paths $q(t) = a(t)q_0$ can be determined the same as uncoupled Keplerian orbits,

$$\begin{aligned} \mathcal{A}_T(a(t)q_0) &= \mathbf{r}^2 \int_0^T \frac{1}{2}\dot{a}^2 + \frac{1}{|a(t)|} \frac{U(q_0)}{\mathbf{r}^2} dt \\ &= 3(2\pi)^{1/3} \left(\tilde{U}(q_0)\right)^{2/3} T^{1/3}. \end{aligned}$$

$$\tilde{U}(q_0) = U\left(\frac{q_0}{\sqrt{I(q_0)}}\right)$$

The **isosceles three body problem** can be described as the special motions of the three body problem whose triangular configurations always describe an isosceles triangle (Wintner (1941)). It is known that this can only occur if two of the masses are the same, and the third mass lies on the symmetry axis described by the binary pair.

Configurations of the isosceles problem require simple constraints: we shall assume that $m_1 = m_2 = m, \quad m_3$ arbitrary,

$$M_{\text{iso}} = \left\{ \mathrm{q} = (r_1, r_2, r_3) \mid r_i \neq r_j, \quad \mathrm{q} \text{ constraints} \right\}.$$

$$\begin{aligned} \sum m_i r_i &= 0, \\ < r_1 - r_2, \mathrm{e}_3 > &= 0, \\ < (r_1 + r_2), \mathrm{e}_i > &= 0, \quad i = 1, 2. \end{aligned}$$

Restrict the potential V to the three dimesional manifold M_{iso}. When all three masses lie in the horizontal plane, the third mass must be at the origin and the three masses are collinear. The set of collinear configurations $\mathcal{S}$ is two dimensional.

$\mathbb{Z}_2$ symmetry across the plane of collinear configurations $\mathcal{S}$ which leaves the potential invariant. We can lift σ to T^*M_{iso} as a **symplectic symmetry** of H, namely

$$\sigma : M_{\text{iso}} \to M_{\text{iso}}, \quad \Sigma(q,p) = (\sigma q, \sigma p).$$

Cylindrical coordinates (r, θ, z) on M_{iso}, where z denotes the vertical height of the mass m_3 above the horizontal plane, and (r, θ) denotes the horizontal position of mass m_1. The symmetry σ takes $(r, \theta, z) \to (r, \theta, -z)$. An elementary argument shows that orbits which cross this plane *orthogonally* are symmetric with respect to σ.

M_{iso} has an $SO(2)$ action which rotates the binary pair around the fixed symmetry axis and leaves $\tilde{V}$ invariant,

$$e^{i\theta}\mathbf{q} = (e^{i\theta}r_1, e^{i\theta}r_2, r_3)$$

. The angular momentum of the system, is a consequence of this action. The rotation action $e^{i\theta}$ is also lifted as a **diagonal symplectic action on T^*M_{iso}**.

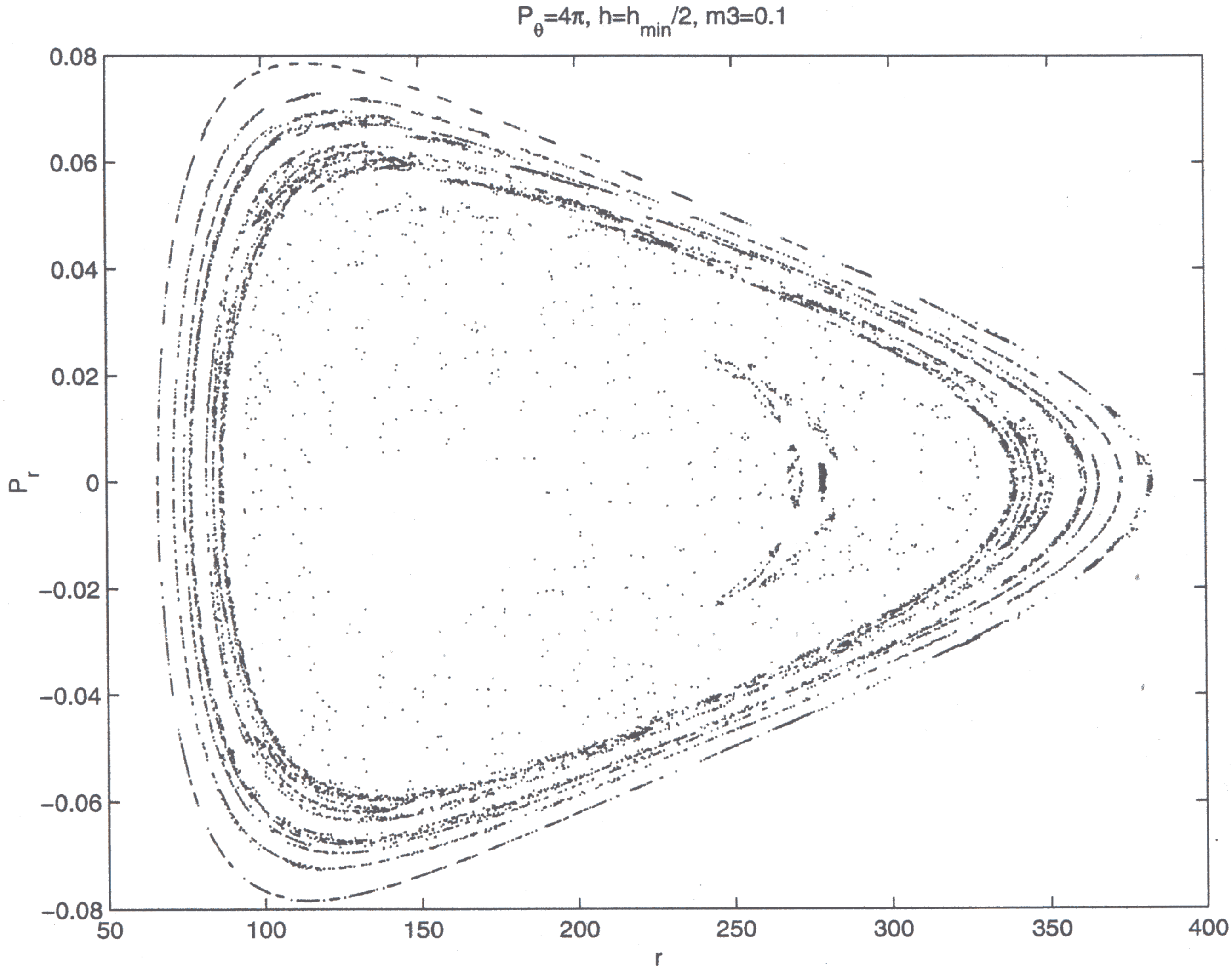
In cylindrical coordinates, the Hamiltonian is

$$\begin{aligned} H &= \frac{p_r^2}{4m} + \frac{p_\theta^2}{4mr^2} + \frac{p_z^2}{2m_3(\frac{m_3}{2m}+1)} \\ &+ -\frac{m^2}{2r} - \frac{2mm_3}{\sqrt{r^2 + z^2(1-\frac{m_3}{2m})^2}}. \end{aligned}$$

The reduced space $\mathbf{J}^{-1}(c)/SO(2)$ has symplectic structure, and reduced Hamiltonian vector field. Substituting $p_\theta = c$ in H gives $H_c = H(r, 0, z, p_r, c, p_z) \ldots$ and corresponding reduced Hamiltonian vector field X_{H_c} on $T^*(M_{\text{iso}}/SO(2))$.

Let $\gamma = (q(t), p(t))$ **denote periodic orbit**. *Jacobi field* along $q(t)$: variation of the configuration $\xi(t)$ which together with the variation in momenta $\eta(t)$, satisfies the equations

$$\frac{d\xi_i}{dt} = m_i \eta_i, \quad \frac{d\eta_i}{dt} = \sum_j \frac{\partial^2 U}{\partial q_i \partial q_j} \xi_j.$$



$P_\theta=4\pi$, $h=h_{min}/2$, m3=0.1
0.08
0.06
0.04
0.02
0
−0.02
−0.04
−0.06
−0.08
P_r
50
100
150
200
250
300
350
400
r

Projection of the reduced space into the reduced configuration space

$$\pi : \mathbf{J}^{-1}(c)/SO(2) \longrightarrow M_{\text{iso}}/SO(2)$$

$\mathcal{L} \subset J^{-1}(c)/SO(2)$ is Lagrangian,

The Maslov cycle is the locus of points $\Sigma \subset \mathcal{L}$ where $d\pi|_{T\mathcal{L}}$ is singular. The *Maslov index* is a generator for the cohomology class on closed curves of Lagrangian planes, and measures the winding number of this curve in the plane of variations transverse to the flow.

We consider the *nonclosed Lagrangian subspaces* of Jacobi fields (ξ, η) which are tangent everywhere to $H_c^{-1}(h)$ within $\mathbf{J}^{-1}(c)/SO(2)$.

Every two dimensional *invariant Lagrangian curve* which is tangent to $H_c^{-1}(h)$ includes the flow direction X_H and one *transverse Jacobi field*, $dH_c((\xi(t), \eta(t)) = 0$,

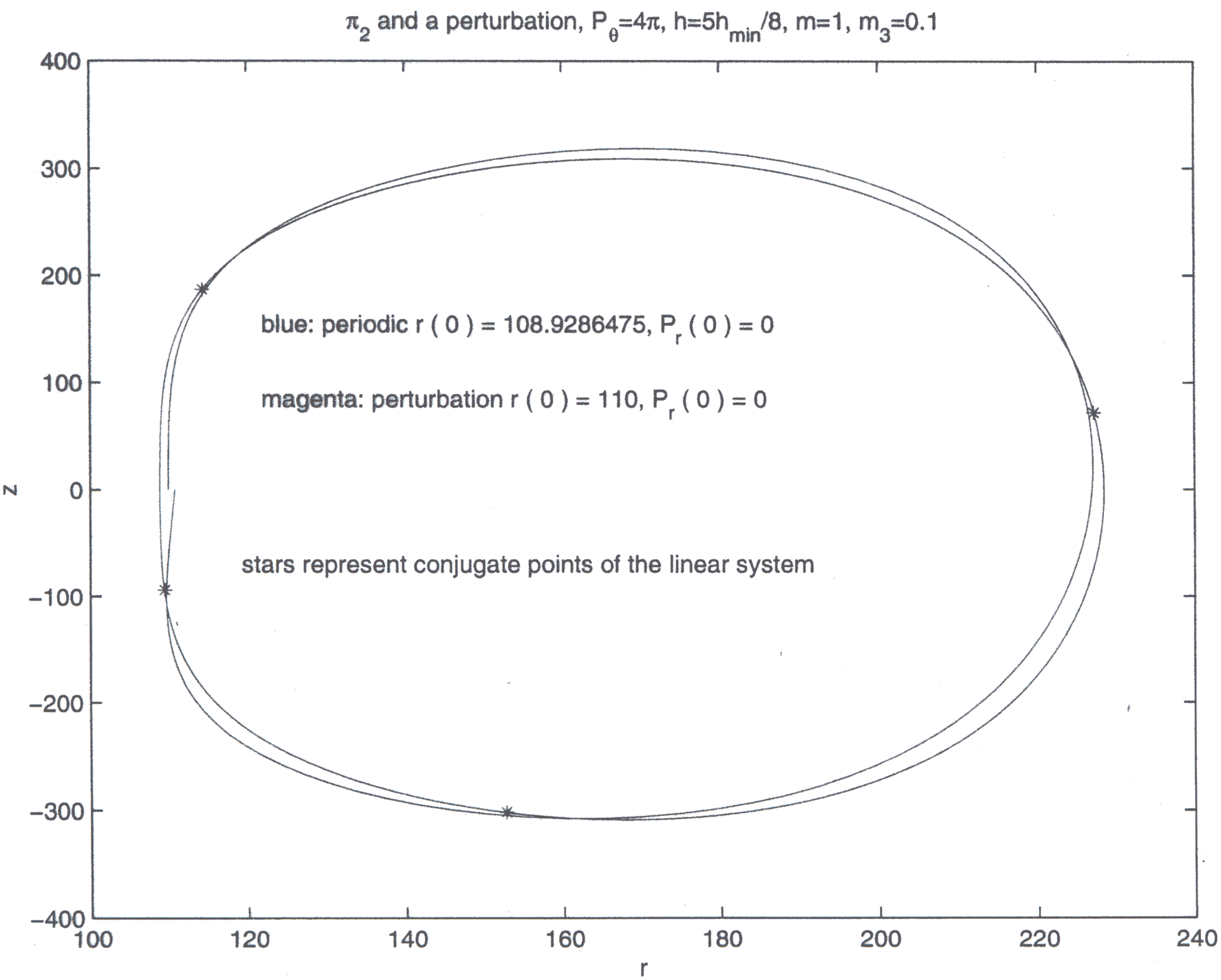
$$\lambda_t = \text{span } \langle(\xi(t), \eta(t)), X_{H_c}(z(t))\rangle.$$

A *focal point* of the Lagrangian plane λ_0 is the value $t = t_0$, where $d\pi : \lambda_{t_0} \to M_{\mathrm{iso}}/SO(2)$ is singular. *Lagrangian singularities along* γ correspond to the vanishing of the determinant

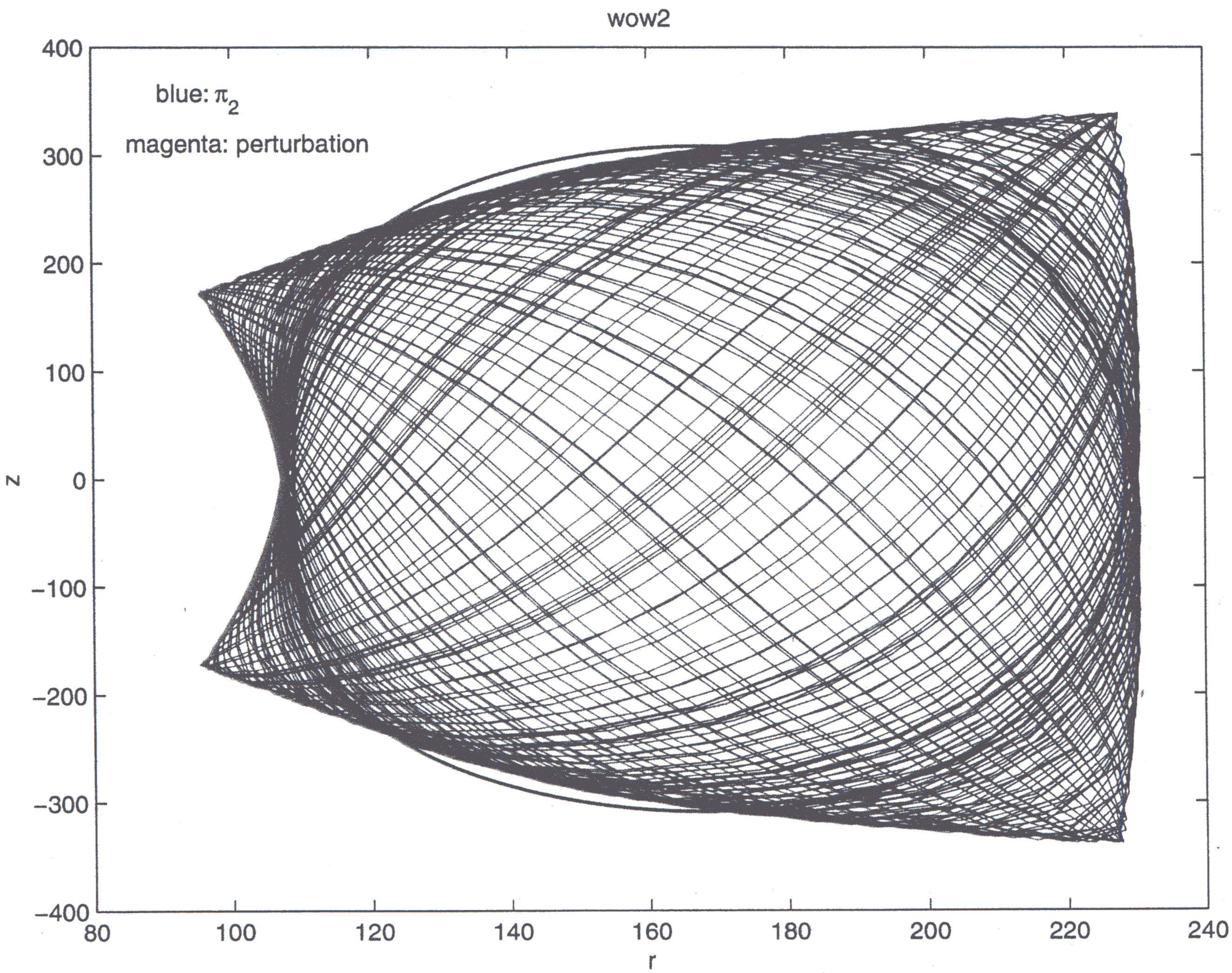
$$D(t_0) = \det \begin{bmatrix} \xi_r(t_0) & p_r(t_0) \\ \xi_z(t_0) & p_z(t_0) \end{bmatrix} = 0, \qquad (4)$$

where (ξ_r, ξ_z) denotes the reduced configuration component of a reduced variational vector field along $(q(t), p(t))$.

Theorem 1. *If γ is a nondegenerate periodic orbit of X_{H_c}, and $i_\gamma = 0$ (mod 2), then $\mathcal{P}$ is hyperbolic without reflection, where $\mathcal{P}$ is the reduced Poincaré map.*



π2 and a perturbation, Pθ=4π, h=5hmin/8, m=1, m3=0.1
blue: periodic r (0) = 108.9286475, Pr (0) = 0
magenta: perturbation r (0) = 110, Pr (0) = 0
stars represent conjugate points of the linear system
400
300
200
100
0
−100
−200
−300
−400
100
120
140
160
180
200
220
240
z
r



wow2
blue: π_2
magenta: perturbation
400
300
200
100
0
−100
−200
−300
−400
z
80
100
120
140
160
180
200
220
240
r

We will look for *symmetric solutions* of the equations of motion. Consider the *fixed time variational problem*

$$\mathcal{A}_T^{\text{iso}}(q) = \int_0^T \sum \frac{1}{2m_i}\|p_{r_i}\|^2 + U(\mathbf{r}) \;\; dt,$$
$$\mathcal{A}_T^{\text{iso}}(x) = \inf_{\Lambda_{\text{iso}}} \mathcal{A}_T(q),$$

$$\Lambda_{\text{iso}} = \left\{ q \in H^1([0,T], M_{\text{iso}}) \mid q(T) = \sigma e^{i\frac{2\pi}{3}} q(0) \right\}.$$

The function space Λ_{iso} contains certain paths which execute rotations and oscillations about the fixed point plane $\mathcal{S}$. $\mathcal{A}_T^{\text{iso}}$ is coercive on Λ_{iso}, using the boundary conditions given. If $q \in \Lambda_{\text{iso}}$ tends to ∞, then the length l of q will also tend to ∞. The solution of the variational problem exists by virtue of **Tonelli's theorem**.

The following inequality is necessary for non-collision (true for all masses)

$$\frac{m^2 + 4mm_3}{3\sqrt{2}} < \left(m^2 + 2mm_3\right)\sqrt{\frac{m+2m_3}{2m+m_3}}$$

True for all m, m_3.

Theorem 2. *A minimizing solution is collision free on the interval $0 \leq t \leq T$, for all choices of masses m, m_3.*

Proof. The argument rests on comparing the collision-ejection homothetic paths which also satisfy the boundary conditions of Λ_{iso}, for three body *central configuration*.

Notice that collinear collision, with a symmetric congruent ejection path rotated by $2\pi/3$, belongs to Λ_{iso} since $\sigma = id$ on $\mathcal{S}$.

Symmetric homothetic equalateral paths will also belong to Λ_{iso}, provided that we insure that the congruent ejection path is rotated by $e^{i2\pi/3}$ so as to satisfy the conditions of Λ_{iso}.

The proof proceeds by comparing the collinear collision-ejection path, with the action of the action of $\frac{1}{3}$ the uniformly rotating collinear *relative equilibrium* having period $3T$. The *relative equilibrium* path has smaller action.

Next, we compare the *relative equilibrium* with the symmetric homothetic equalateral path...

Neither the collinear homothetic, nor the equalateral homothetic path in Λ_{iso} provides a global minimizer for the action.

The *relative equilibrium* solution is not minimizing ....

Theorem 3. *The solution $q(t)$ to the variational problem may be extended so as to satisfy the relation $q(t+T) = \sigma e^{i\frac{2\pi}{3}} q(t)$. The corresponding momentum $p(t)$ satisfies the same symmetry $p(t+T) = \sigma e^{i\frac{2\pi}{3}} p(t)$. Together, the pair $(q(t), p(t))$ may be extended to a $6T$ periodic orbit of the Hamiltonian vector field X_H for isosceles Hamiltonian which undergoes two full rotations and six oscillations in each period, and which is not the collinear* relative equilibrium *in $\mathcal{S}$.*

For given integers (M, N) we can study the more general variational problem

$$\begin{aligned} \mathcal{A}_T^{\text{iso}}(q) &= \int_0^T \sum \frac{1}{2m_i} \|p_{r_i}\|^2 + U(\mathbf{r}) dt \\ \mathcal{A}_T^{\text{iso}}(x) &= \inf_{\Lambda_{(M,N)}} \mathcal{A}_T(q), \end{aligned}$$

$$\Lambda_{(M,N)} = \left\{ q \in H^1([0,T], M_{\text{iso}}) \mid q(T) = \sigma e^{i\frac{2M\pi}{N}} q(0) \right\}.$$

Theorem 4. *The solution $q(t)$ to the variational problem is collision free on the interval $[0, T]$ provided that the inequality $M\tilde{U}_1 < N\tilde{U}_0$. This occurs in the equal mass case, provided that $M < \frac{3\sqrt{2}}{5} N$, and in the case when $m_3 = 0$ when $M < N$. The solution $q(t)$ may be extended so as to satisfy the condition $q(t+T) = \sigma e^{i\frac{2M\pi}{N}} q(t)$, and together with $p(t)$ gives a NT-periodic integral curve of in case N is even, and a $2NT$-periodic integral curve in case N is odd.*

Theorem 5. *The σ-symmetric orbit which extends the solution $q(t)$ to the variational problem (M,N) is unstable, and when projected to reduced space, is hyperbolic whenever nondegenerate.*

We sketch the idea of the proof.

The functional and its differentials evaluated in the direction $\xi \in T_{q(t)}\Lambda_{(M,N)}$ along a critical curve $q(t)$ are

$$
\begin{aligned}
\mathcal{A}_T^{\text{iso}}(q) &= \int_0^T \sum \frac{1}{2m_i}\|p_{r_i}\|^2 + U(\mathbf{r})dt, \\
\delta\mathcal{A}_T^{\text{iso}}(q)\cdot\xi &= \sum_i \langle p_{r_i}, \xi_i\rangle|_0^T \\
&+ \int_0^T \sum_i \langle -\frac{d}{dt}p_{r_i} + \frac{\partial U}{\partial q_i}, \xi_i\rangle dt, \\
\delta^2\mathcal{A}_T^{\text{iso}}(q)(\xi,\xi) &= \sum_i \langle \eta_i, \xi_i\rangle|_0^T \\
&+ \int_0^T \sum_{i,j} \langle -\frac{d}{dt}\eta_i + \frac{\partial^2 U}{\partial q_i \partial q_j}\xi_j, \xi_i\rangle dt.
\end{aligned}
$$

We focus on the Jacobi fields $\xi(t)$ which satisfy the boundary relation $\xi(T) = \sigma e^{i\frac{2M\pi}{N}}\xi(0)$. The Jacobi fields (ξ, η) may be projected to the reduced space $\mathbf{J}^{-1}(c)/SO(2)$.

An essential component of the second variation, for Jacobi fields in $T_{q(t)}\Lambda_{(M,N)}$ which together with the conjugate variations $\eta(t)$ whose projections are everywhere tangent to the energy surface $H_c^{-1}(h) \subset \mathbf{J}^{-1}(c)/SO(2)$.

Now we study the **reduced Lagrangian curve** λ_t^* of reduced energy-momentum tangent variations

$$\lambda_0^* = \{(\xi(0), \eta(0)) | \xi(T) = \sigma\xi(0), \quad dH_c(\xi(0), \eta(0)) = 0$$

Evidently, the subspace λ_0^* is not empty, since $X_{H_c}(q(0), p(0)) \in \lambda_0^*$.

Lemma 1. *If the periodic integral curve $(q(t), p(t))$ is nondegenerate on the reduced energy-momentum manifold $H^{-1}(c)$ within $\mathbf{J}^{-1}(c)/SO(2)$, then λ_0^* is Lagrangian.*

The *second order necessary conditions* in terms of reduced energy-momentum variations, in order that $q(t)$ is a minimizing solution of the variational problem.

Proposition 1. *If the curve $q(t)$ is a collision-free solution of the variational problem, and is nondegenerate as a periodic integral curve of X_{H_c} then λ_0^* has no focal points in the interval $[0, T]$, and*

$$\omega(\lambda_0^*, \lambda_T^*) > 0.$$

The description of the stability properties of the minimizing curves now proceeds from a comparison of the Lagrange planes λ_0^*, λ_T^*.

Lemma 2. *The Lagrange planes $(\sigma\mathcal{P})^n\lambda_0^*$ have no focal points in the interval $0 \le t \le T$.*

$\sigma\mathcal{P}\lambda_0^*$ is focal point free on $[0, T]$ so that comparing with $(\sigma\mathcal{P})^2\lambda_0^*$ and using the orientation supplied by the symplectic form ω indicates that $(\sigma\mathcal{P})^2\lambda_0^*$ is focal point free on the interval $[0, T]$ as well.

Lemma 3. *The reduced Lagrange plane* λ_0^* *of transverse variations is focal point free on the interval* $0 \leq t < \infty$.

Since there is no rotation of the Lagrange planes λ_t^*, we can ask what obstruction there is to prevent this. The answer given in the next Theorem, is that the Poincaré map must have real invariant subspaces.

Theorem 6. *Under the assumptions of Proposition (1) , there are (real) invariant subspaces for the reduced Poincaré map* $\mathcal{P}^2$. *These subspaces are transverse when* $\delta^2 \mathcal{A}_T(q)$ *is nondegenerate when restricted to the subspace in the reduced space of tangential variations* $T_{q(t)}\Lambda_{(M,N)}$ *which are also tangent to the reduced energy surface* $H_c^{-1}(h)$.

The proof proceeds by examining the iterates $(\sigma\mathcal{P})^n\lambda_0^*$ of the subspace λ_0^* of tangential Jacobi variations in the reduced energy momentum space $H_c^{-1}(h)$.

No focal points, and the fact that
$\omega((\sigma\mathcal{P})^n\lambda_0^*, (\sigma\mathcal{P})^{n+1}\lambda_0^*) > 0$,
implies that the iterates $(\sigma\mathcal{P})^n\lambda_0$ must have a limit subspace $\beta = \lim_{n\to\infty}(\sigma\mathcal{P})^n\lambda_0^*$.

The subspace β is thereby Lagrangian, and invariant for the symplectic map $\sigma\mathcal{P}$. Therefore this implies that since $\sigma\mathcal{P} = \mathcal{P}\sigma$,

$$\begin{aligned} \sigma\mathcal{P}\beta &= \beta \\ \mathcal{P}\beta &= \sigma\beta \\ \mathcal{P}^2\beta &= \mathcal{P}\sigma\beta \\ &= \sigma\mathcal{P}\beta \\ &= \beta. \end{aligned}$$

It is not difficult to see that the subspace β can be also represented as the forward limit of the iterates $\mathcal{P}^n$ of the vertical space, $\beta = \lim_{n\to\infty}\mathcal{P}^n V|_{(x,p)}$. It follows that β represents the reduced transverse directions of the *stable manifold* of (q,p).

Similarly the *unstable manifold* α may be represented as the limit in backwards time ,
$\alpha = \lim_{n\to\infty}\mathcal{P}^{-n}V|_{(x,p)}$.

Finally, to show transversality of the subspaces β, α we can use the fact that in case (q, p) is nondegenerate, $\omega(\lambda_0^*, \sigma\mathcal{P}\lambda_0^*) > 0$, by virtue of Proposition (1). This implies that in the case of nondegeneracy $\omega(\alpha, \beta) > 0$, which implies transversality. This concludes the proof.