Mirror Principle

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Outline

- Overview of problems
- Mirror Principle
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- Some Conjectures

Overview

- Topological sigma model
  = intersection theory on complex loop space.

- "Complex loop space" of a projective manifold X:

$$\{f : \Sigma \to X \ \ holo.\}$$

- Fix $f_*[\Sigma] = d \in H_2(X, \mathbf{Z})$, $genus(\Sigma) = g$; but allow Σ to vary, and decorate Σ by finitely many points $p_1, .., p_k$. The mapping space is a finite dimensional quasi-projective variety.

- Problem: Do intersection theory on (modified version of) this mapping space.

• Naive approach:

• Mapping space is a quasi-projective variety

$$M_{g,k}(d, X) = \{(\Sigma, f, x_1, .., x_k)\}$$

with expected dimension, say R.

• Incidence conditions: fix cycles $V_1, .., V_k$ in X with

$$\Sigma \, codim \; V_i = R$$

and require that

$$f(x_i) \in V_i.$$

• $\{(\Sigma, f, x_1, .., x_k) | f(x_i) \in V_i\}$ should have dimension 0. Regarded as a 0-cycle, its degree would be number:

$$(V_1, .., V_k) \mapsto a\ number$$

BUT...

• $M_{g,k}(d, X)$ is noncompact and typically has the wrong dimension.

• The incidence conditions need not cut down to 0 dimension.

• Ruan-Tian (symplectic), Kontsevich (algebraic): formulate intersection theory on compactified mapping spaces.

• Stable map moduli space:

$$\bar{M}_{g,k}(d, X) := \{(C, f, x_1, .., x_k)\}/\sim$$

where C is a genus g projective curve, at worst nodal. $f : C \to X$ is a degree d map, and $x_1, .., x_k$ are smooth points on C.

• Stability condition:
if $f(C_1) = pt$ then C_1, together with its special points, has no infinitesimal auto.

• Equiv. relation:
$(C, f, x_1, .., x_k) \sim (C', f', x'_1, .., x'_k)$ if there is an isomorphism h

$$\begin{array}{ccc} x_i & \mapsto & x'_i \\ C & \xrightarrow{h} & C' \\ f \searrow & \circ & \swarrow f' \\ & X & \end{array}$$

• $\bar{M}_{g,k}(d, X)$ can have impure dimension. Li-Tian construct a cycle in Chow group $A_R(\bar{M}_{g,k}(d, X))$ (cf. Fukayo-Ono, Behrend-Fentachi, Ruan, Siebert): virtual fundamental cycle for $\bar{M}_{g,k}(d, X)$.

• Notation: $LT_{g,k}(d, X)$ be the virtual fundamental cycle of $\bar{M}_{g,k}(d, X)$ of pure dimension

$$R = \langle c_1(X), d\rangle + (1 - g)dim(X) + k - 3.$$

• It plays the role of the fundamental cycle of a compact manifold.

Problem

• Fix a vector bundle E on $M_{g,k}(d, X)$, and a char. class $b(E) \in A^*(M_{g,k}(d, X))$. Fix cohomology classes $\omega_1, .., \omega_k$ on X. Study the integrals

$$K_D := \int_{LT_{g,k}(d,X)} e_1^*\omega_1 \cdots e_k^*\omega_k \, b(E).$$

$D = (g, k; d)$.

• For simplicity, will restrict to $\omega_1 = \cdots = \omega_k = 1$. All results here have been generalized to the case when ω_i are arbitrary. The class b will be Euler class, Chern polynomial, or more generally any multiplicative class.

• Definition: A vector bundle $V \longrightarrow X$ is called *convex* if $H^1(\mathbf{P}^1, f^*V) = 0$ for any holomorphic map $f : \mathbf{P}^1 \longrightarrow X$.

• A convex bundle induces

$$\begin{array}{ccc} V_d & \qquad\qquad & H^0(C, f^*V) \\ \downarrow & & \downarrow \\ M_{0,k}(d, X) & & (C, f). \end{array}$$

• Examples: the tangent bundle of $X = \mathbf{P}^n$; any positive power of the hyperplane bundle.

• Similarly for concave bundle V: $H^0(C, f^*V) = 0$, $\forall f : C \longrightarrow X$ genus g maps.

• Denote by $E = V_D \longrightarrow M_{g,k}(d, X)$, $D = (g, k; d)$, the vector bundle induced by a convex/concave bundle V. Also write $V'_D \longrightarrow M_{g,k+1}(d, X)$.

The Gluing Identity

• Enlarge $M_{g,k}(d, X)$ to

$$M_D := M_{g,k}((1, d), \mathbf{P}^1 \times X).$$

The projection $\mathbf{P}^1 \times X \to X$ induces a map

$$M_D \xrightarrow{\pi} M_{g,k}(d, X).$$

Pulling back $b(V_D)$ via π, we get a cohomology class $\pi^* b(V_D)$ on M_D.

• $\mathbf{C}^\times$ acts on $\mathbf{P}^1$ by the standard rotation. This induces an $\mathbf{C}^\times$ action on M_D. Will do localization on M_D relative to this action.

• Each fixed point in M_D comes from gluing pairs in $M_{g_1,k_1+1}(d_1, X) \times M_{g_2,k_2+1}(d_2, X)$ at a marked point x. Here $D = D_1 + D_2$ where $D_i = (g_i, k_i; d_i)$.

• Call this component F_{D_1,D_2}, and $i : F_{D_1,D_2} \to M_D$ inclusion. There are two natural projection maps

$$p_0 : F_{D_1,D_2} \to M_{g_1,k_1+1}(d_1, X)$$
$$p_\infty : F_{D_1,D_2} \to M_{g_1,k_1+1}(d_1, X)$$

Pulling back $b(V'_{D_1})$ via p_0, and $b(V'_{D_2})$ via p_∞, we get cohomology classes $p_0^* b(V'_{D_1})$ and $p_\infty^* b(V'_{D_2})$ on F_{D_1,D_2}.

• Theorem(Gluing Identity): On F_{D_1,D_2} we have identity of cohomology classes:

$$i^* \pi^* b(V_D) = p_0^* b(V'_{D_1})\, p_\infty^* b(V'_{D_2}).$$

• Next: transfer this identity to some simple manifold...

Functorial localization

• Given $f : A \to B$, a G-equiv. map of G manifolds;

$$\begin{array}{cccc} f^{-1}(E) \supset & F & \stackrel{i_F}{\to} & A \\ & g\downarrow & & \downarrow f \\ & E & \stackrel{j_E}{\to} & B. \end{array}$$

For $\omega \in H_G^*(A)$, we have identity on E:

$$\frac{j_E^* f_*(\omega)}{e_G(E/B)} = g_* \frac{i_F^*(\omega)}{e_G(F/A)}.$$

Comparison theorem

• There is a version for stable map moduli:

$$i : F_{D_1,D_2} \to M_D$$

plays the role of $i_F : F \to A$. Evaluation map

$$e : F_{D_1,D_2} \to X$$

evaluating at gluing point plays the role of $g : F \to E$.

• Fix a projective embedding $X \subset \mathbf{P}^n$. Each map stable $(f, C, x_1, .., x_k)$ is a degree $(d, 1)$ map into $X \times \mathbf{P}^1 \subset \mathbf{P}^n \times \mathbf{P}^1$:

• Corresponding to this are $n+1$ polynomials $f_i(w_0, w_1)$ each vanishing of order d_i at $[a_i, b_i] \in \mathbf{P}^1$.

• Theorem(Li-Lian-Liu-Yau): The corrrespondence

$$(f, C, x_1, .., x_k) \mapsto [f_0, .., f_n]$$

defines an equivariant morphism $\varphi : M_D \to N_d$ where N_d is the projective space of $(n+1)$-tuple of polynomials of degree d.

• The fixed points in N_d are copies of $\mathbf{P}^n$. There is a similar theorem if we have an embedding $X \subset \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_m}$. Then N_d is replaced by a product W_d of N_d's. Label the fixed points by Y_{d_1,d_2}, and inclusion

$$j : X \subset Y_{d_1,d_2} \to W_d.$$

• Putting together a commutative square:

$$\begin{array}{ccc} F_{D_1,D_2} & \xrightarrow{i} & M_D \\ e \downarrow & \circ & \downarrow \varphi \\ X & \xrightarrow{j} & W_d. \end{array}$$

• <u>Theorem</u>: (Comparison Theorem) For any equivariant class ω on M_D, we have an identity on X:

$$\frac{j^*\varphi_*(\omega \cap LT_D)}{e(X/W_d)} = e_* \frac{i^*\omega \cap [F_{D_1,D_2}]}{e(F_{D_1,D_2}/M_D)}.$$

Denote the RHS by $J_{D_1,D_2}\omega$.

• <u>Theorem</u>: Consider the integral

$$K_D = \int_{LT_{g,k}(d,X)} b(V_D).$$

Suppose the integrand has the right degree. Then

$$\int_X e^{-H\cdot t} J_{O,D}\pi^* b(V_D) = (-1)^g(2-2g-d\cdot t)K_D.$$

• Thus the goal is to compute the numbers K_D by first computing the classes $J_{D_1,D_2}\pi^* b(V_D)$ on X. Let's restrict to $g=0$ and $k=0$ for simplicity.

Solving the Gluing Identity

• Gluing Identity $\Longrightarrow$

• Theorem: We have the identity of cohomology classes on X:

$$
\begin{aligned}
& b(V) \cdot J_{D_1,D_2} \pi^* b(V_D) \\
&= J_{D_1,O} \pi^* b(V_{D_1}) \cdot J_{O,D_2} \pi^* b(V_{D_2}).
\end{aligned}
$$

• For general X, complete classification of solutions not available.

• Important Fact: the Gluing Identity is functorial; if $V \to X$ is T-equivariant bundle, there is a T-equivariant version.

• Definition: A T-manifold X is called a balloon manifold if

i. X^T is finite

ii. (GKM) T-weights on T_pX at fixed point p are pairwise linearly independent.

iii. The moment map is injective on X^T.

• Examples: projective toric manifolds, flag manifolds.

• For ANY balloon manifold X, the T-equiv. Gluing Identity can be solved completely in terms of restrictions $TX|_C$ and $V|_C$ where $C \cong \mathbf{P}^1$ are T-invariant curves in X.

• There is a linear algorithm to compute all equivariant classes $J_{D_1,D_2}\pi^* b(V_D)$, hence all intersection numbers K_D, in terms of these data.

• Example: X: toric manifold
$D_1, .., D_N$: T-invariant divisors
$V = \oplus_i L_i, \quad c_1(L_i) \geq 0$ and $c_1(X) = c_1(V)$.
$b(V) = e(V)$

$$\Phi(T) = \Sigma K_D e^{d \cdot T}.$$

$$B(t) = e^{-H \cdot t} \sum_d \prod_i \prod_{k=0}^{\langle c_1(L_i), d\rangle} (c_1(L_i) - k)$$

$$\times \frac{\Pi_{\langle D_a, d\rangle < 0} \Pi_{k=0}^{-\langle D_a, d\rangle - 1} (D_a + k)}{\Pi_{\langle D_a, d\rangle \geq 0} \Pi_{k=1}^{\langle D_a, d\rangle} (D_a - k)} e^{d \cdot t}.$$

• Computing generating function $\Phi(t) = \Sigma K_d e^{dt}$. There are explicitly computable functions $f(t), g(t)$, such that

$$\int_X \left(e^f B(t) - e^{-H \cdot T} e(V)\right) = 2\Phi - \Sigma T_i \frac{\partial \Phi}{\partial T_i}$$

where $T = t + g(t)$ (mirror transformation).

Mirror History

- **PHASE I**:

- Gepner, Lerche-Vafa-Warner, Dixon (mid 80): idea of mirror conformal field theories.

- Greene-Plesser, Candelas-Lynker-Schimrigk, Klemm,. 89): mirror CYs in weight projective spaces.

- Candelas-de la Ossa-Green-Parkes (90): use mirror CYs to give enumerative predictions for quintics.

- Libgober-Teiteilboim, Morrison, Batyrev, Klemm et al, Candelas et al, Berglund et al, Hosono et al, ...(91-93): enumerative predictions for many examples of weighted projective complete intersection CYs.

- Batyrev, Borisov (91-93): mirror CYs in toric varieties.

- Hosono-Lian-Yau (94): propose genus 0 mirror formula for general toric CY complete intersections.

- Bershadsky-Cecotti-Ooguri-Vafa (95): higher genus formula.

• **PHASE II**:

• Vafa, Witten, Kontsevich, Ruan-Tian: math. foundation of quantum cohomology and intersection numbers.

• Ellingsrud-Stromme, Kontsevich (94): apply directly Atiyah-Bott to genus-0 Euler class of Candelas et al for $\mathbf{P}^4$.

• Givental, Bini-de Concini-Polito-Procesi, Pandharipande, (96-98): apply Atiyah-Bott and quantum cohomology theory to genus-0 Euler class for $\mathbf{P}^n$.

• Lian-Liu-Yau (97): develop functorial localization to any multiplicative char. classes, and new genus-0 formulas for $\mathbf{P}^n$.

• Klemm, Katz, Mayr, Vafa,..(97): B-model local mirror symmetry.

• Lian-Liu-Yau (97): math. foundation for A-model local mirror symmetry.

• **PHASE III**:

• Li-Tian, Behrend-Fantachi,... (97): foundation for virtual fundamental cycles.

• Graber-Pandaripande (97): Virtual localization.

• Li-Tian (98): symplectic and algebraic quantum cohomology theories are equivalent.

• Lian-Liu-Yau (98-99): apply functorial localization to any multiplicative classes for any projective manifold, at higher genus.

• Lian-C.H.Liu-Yau (99): reconstruct multiplicative classes for hypersurfaces of general type without mirror formula.

• Most recently: functorial localization of Lian-Liu-Yau becomes a popular technique. Eg. Bertram, Lee, ... cf. Gathmann.

Conjectures

• Let Y be a CY 3-fold. Then the virtual class $LT_{0,0}(d,Y)$ is a zero dimensional cycle. Let $K_d \in \mathbf{Q}$ be the degree of this cycle, and define the "instanton numbers" n_d by the formula

$$K_d = \sum_{k|d} n_{d/k}.$$

• Conjecture 1: the n_d are all integers.

• When Y is a toric complete intersection, then the n_d should be divisible by the "multidegrees" of Y.

• Example: when Y is the quintic 3-fold, the n_d are divisible by 5^3 (Clemens). Verified by Lian-Yau for $5 \not| d$.

• Near the "large radius limit", the periods of the mirror manifold X should be of the form, in local coordinates

$$\omega_0 = 1 + O(z), \qquad \omega_i = \omega_0 log\ z_i + O(z), \qquad \cdots$$

The mirror map $z \mapsto q$ has power series

$$q_i := exp(\frac{\omega_i}{\omega_0}) = z_i + O(z^2).$$

• Conjecture 2: The expansions of the q_i have integral coefficients.

• This has been verified by Lian-Yau for hypersurfaces X in toric varieties with $H^2(X, \mathbf{Z}) = \mathbf{Z}$.

• When X is a toric complete intersections, the series q_i^{1/h_i} should also have integer expansion, where h_i are "multidegrees" of Y.

• Example: when X is the mirror quintic, $h = 5$, this has been observed by Vafa et al.