# Pricing in markets modeled by general processes with independent increments

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# Agenda

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# Rationale for jump-diffusion modeling

## Properties of real-life financial time series not reflected in the Black–Scholes model

### A Nonstationarity

- real markets change qualitatively over time
- calibration of parameters to historical data is suspect
- "regime–switching" models

### B Volatility clustering

- squared returns are serially correlated
- leads to ARCH/GARCH/stochastic volatility models

## C Heavy tailed distributions

- increased probabilities for "large moves/extreme events"
- underlying noise should have non–gaussian heavy tails

## D Multivariate dependences

- dependence structure of "large moves" may be quite poorly predicted by the covariance
- need flexibility to model large moves differently from "normal market moves"

**Jump diffusion modeling addresses C and D**

# Jump diffusion modeling setup

Market: assumed "efficient" and "frictionless"

- riskless asset: $dB_t = rB_t dt, \quad 0 \leq t \leq T$

$$\text{take } r = 0, B \equiv 1$$

- $N$ risky assets:

$$dS_t^i = S_t^i \quad \left[\mu^i dt + \sum_{a=1}^{M} \sigma^{ia} dW_t^a \qquad \right]$$

**Remark:** Diffusion processes are continuous at all times, almost surely.
Add in **JUMP TERM** $\quad S_{t^-}^i dQ_t^i$

$$\left( S_{t^-}^i = \lim_{\tau \uparrow t} S_\tau^i, \quad S_{t^+}^i = \lim_{\tau \downarrow t} S_\tau^i \right)$$

Log returns: let $s_t^i = \log S_t^i$

$$ds_t^i = \left[\mu^i - \frac{1}{2}(\sigma\sigma^T)^{ii}\right] dt + \sum_{a=1}^{M} \sigma^{ia} dW_t^a$$
$$+ \int_{\mathbb{R}^N} z^{(i)} \; N_t^{(\nu)}(dt \; d^N z)$$

Poisson random measure $N^{(\nu)}$:
For any set $(t_1, t_2] \times A \subset \mathbb{R}^+ \times \mathbb{R}^N$

$N_t^{(\nu)}\Big((t_1, t_2] \times A\Big)$ = number of jumps $s_{t+} - s_{t-}$ of log return vector which lie in $A$, which occur in time interval $(t_1, t_2]$
= Poisson random variable with intensity parameter $\lambda\Big((t_1, t_2] \times A\Big) = |t_2 - t_1| \nu(A)$

intensity measure

$N_t^{(\nu)}$ is a Poisson Point Process

# Generalized Ito Formula

If $F : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^M$ is twice differentiable and $X_t$ is an $\mathbb{R}^N$–valued jump diffusion with $dX_t = dX_t^{(cts)} + dX_t^{(jump)}$ then $F(t, X_t)$ is an $\mathbb{R}^M$–valued jump diffusion and

$$\begin{aligned} dF_t &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX_t^{(cts)} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}d\left\langle X, X\right\rangle_t^{(cts)} \\ &+ \int_{\mathbb{R}^N} \left[F(t, X_{t^-} + z) - F(t, X_{t^-})\right] \; N_t^{(\nu)}(dt \; d^N z) \end{aligned}$$

**Example:** $S_t = \exp[s_t]$

$$\begin{aligned} dS_t &= S_t\left[\left(\mu - \frac{1}{2}(\sigma\sigma^T)\right)dt + \sigma dW_t^a\right] \\ &+ S_t\sigma^2 dW_t dW_t \\ &+ \int_{\mathbb{R}^N} \left[\exp[s_{t^-} + z] - \exp[s_{t^-}]\right] \; N_t^{(\nu)}(dt \; d^N z) \end{aligned}$$

$$\therefore \qquad dQ_t^{(i)} = \int_{\mathbb{R}^N} \left[e^{z^i} - 1\right] \; N_t^{(\nu)}(dt \; d^N z)$$

Facts:

- jump diffusion markets are **INCOMPLETE**

- in incomplete markets "risk–neutral pricing theory" (Black–Scholes et al) must be replaced by "optimal portfolio theory"

# The optimal portfolio problem

An economic agent invests in market over $[0, T]$ creating a portfolio with value $X_t$, so as to maximize $E(U(X_T))$, the "expected utility of terminal wealth"

<u>Utility:</u> function $U : \mathbb{R} \to [-\infty, \infty)$ satisfying

(i) monotonically increasing

(ii) strictly concave

$U(x) =$ "pleasure" derived from having \$$x$ at $T$

Portfolio strategy $\pi$:

- At each time $t$, the agent has wealth $X_t$
- chooses to invest $\pi_t^{(i)}$ in stock $i$

$$\therefore \qquad X_t = \sum_{i=1}^{N} \pi_t^{(i)} + (X_t - \sum_{i=1}^{N} \pi_t^{(i)})$$

stocks          bank account

Self financing condition: No $ put in or taken out

$$\begin{aligned} dX_t &= \sum_{i=1}^{N} \pi_{t^-}^{(i)} \left( \frac{dS_t^{(i)}}{S_{t^-}^{(i)}} \right) + 0 \\ &= \sum_{i=1}^{N} \pi_{t^-}^{(i)} \left( \mu^i dt + \sum_{a=1}^{M} \sigma^{ia} dW_t^a + dQ_t^{(i)} \right) \end{aligned}$$

**Optimization for agent with utility $U$**

For each value of the initial wealth $x$ find the pair $(u(x), \pi^*(x))$ which optimize

$$u(x) = \sup_{\pi} E\Big(U(X_T^{\pi,x})\Big)$$

$u(x) =$ value function
$\pi^* =$ optimal strategy

# Option pricing by Davis' "marginal rate of substitution"

Let $F_T$ be contingent claim with expiry date $T$

Q How to assign a value $F_0$?

A For an agent with utility $U$ and wealth $x$:

$$F_0(U,x) = \frac{E(U'(X_T^{\pi^*,x})F_T)}{E(U'(X_T^{\pi^*,x}))}$$

Logic

- For $\epsilon$ (small) at $t=0$ invest $\epsilon$ in the option, and remainder in the optimal portfolio

$$x = (x-\epsilon) + \epsilon$$

portfolio option

- for $0 < t < T$ adopt the optimal strategy $\pi^*(x-\epsilon)$
- at $t=T$, $X_T^\epsilon = (X_T^{\pi^*,x} - \epsilon) + \epsilon(F_T/F_0)$
- $F_0$ determined by $E(U(X_T^\epsilon) = E(U(X_T^0)) + \mathcal{O}(\epsilon^2)$

# An example

- take a general JD market with constant $(\mu, \sigma, \nu)$

- $U(x) = -e^{-\alpha x}$,
  $\alpha > 0$ constant

- solve the optimal problem using the Hamilton–Jacobi–Bellman equation from stochastic control

Verification Theorem

Suppose $H(t,x), g(t,x)$ are such that

1. $H$ is sufficiently integrable and solves

$$\begin{cases} \frac{\partial H}{\partial t} + & \sup_\pi \Bigg[ (\pi \cdot \mu) \frac{\partial H}{\partial x} + \frac{1}{2} |\sigma^T \pi|^2 \frac{\partial^2 H}{\partial x^2} \\ + & \int_{\mathbb{R}^N} \left[ H(t, x + \pi \cdot (e^z - 1)) - H(t,x) \right] \nu(d^N z) \Bigg] \\ & = 0 \\ H(T,x) & = U(x) \qquad \forall x \in \mathbb{R} \end{cases}$$

2. the sup is achieved by $\pi^{(i)} = g^{(i)}(t,x)$

Then:

1. The value function is $u(x) = H(0,x)$

2. The optimal strategy exists and is given by $\pi_t^* = g(t, X_t)$.

Assume $H(t,x) = -f(t)e^{-\alpha x}$, $f(t) > 0$
Find the condition for optimal $\pi$ is independent of $t, x$:

$$\sup_{\pi}\Big[\alpha(\pi\cdot\mu) - \frac{\alpha^2}{2}|\sigma^T\pi|^2 - \int_{\mathbb{R}^N}\left[e^{-\alpha\pi\cdot(e^z-1)} - 1\right]\nu(d^N z)\Big]$$

Result:

- last two terms are strictly concave, hence optimal strategy $\pi^*$ exists
- $\alpha\pi^*$ is independent of $\alpha, t, x$
- "constant value in each risky asset"
- Value function is $u(t,x) = -e^{K(T-t)-\alpha x}$ ($K$ constant).

# Duality Theory

(Kar-Leh-Shr-Xu 91)
(Kram-Sch 99)

Introduce the Legendre transform

$$\begin{cases} V(y) = \tilde{U}(y) = \sup_{x\in\mathbb{R}}[U(x) - xy] \\ \\ U(x) = -(\widetilde{-V})(x) = \inf_{y\in\mathbb{R}}[V(y) + xy] \end{cases}$$

Similarly for the value function:

$$v(y) = \tilde{u}(y) \longleftrightarrow u(x)$$

**Example:** (cont'd)
For $u(t,x) = -e^{K(T-t)-\alpha x}$

$$v(t,y) = (y/\alpha)\Big(\log(ye^{K(t-T)}/\alpha) - 1\Big)$$

**Theorem 1.** *(KS '99)* *Assume:*

- *general semi–martingale market*
- *smoothness and growth conditions on $U$*

*Then:*

1. $v(y)$ *solves a dual optimal problem:*

$$v(y) = \inf_{Y \in \mathcal{Y}(y)} E(V(Y_T))$$

*where*

$$\mathcal{Y}(y) = \{Y_t > 0 | X_t Y_t \text{ supermartingale } \forall \text{ portfolios } X\}$$

2. *optimizers* $\hat{X}(x)$ *and* $\hat{Y}(y)$ *exist and are related by*

$$\hat{X}(x) = -V'(\hat{Y}(y)); \ \hat{Y}(y) = U'(\hat{X}(x))$$

*where* $x = -v'(y), y = u'(x)$*.*

Option pricing:

$$\begin{aligned} F_0(U,x) &= \frac{E(\widehat{Y}(y)F_T)}{E(\widehat{Y}(y))} \\ &= E([\widehat{Y}(y)/y]F_T) \\ &= E_{\widehat{Q}(y)}(F_T) \end{aligned}$$

where $x = -v'(y)$.

$\frac{d\widehat{Q}(y)}{dP} = \widehat{Y}(y)/y \equiv$ equivalent martingale measure

**Example** (cont'd)

- dual value function $v(t, y)$ indeed solves the constrained dual HJB equation, confirming the KS theorem in this case

- $\widehat{Y}(y)/y$ is independent of $y$ and coincides with Schweizer's "minimal martingale measure"

# Conclusions

- "incomplete markets" are those for which Card$\Big(\mathcal{Y}(y)\Big) > 1$
  - then dual problem is nontrivial
  - not all contingent claims can be hedged, or priced uniquely

- even simple jump diffusion models are massively incomplete, and resulting HJB equations are complicated

- the theory of jump diffusion markets exists and is developing rapidly

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